Module 1 – Electromagnetic field propagation at material interfaces

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Introduction

A plane electromagnetic wave is specified by its temporal frequency ω, a propagation vector $\boldsymbol{k}$, an electric-field amplitude $\boldsymbol{E}_o$, and a magnetic-field amplitude $\boldsymbol{H}_o$. In general, electromagnetic fields are real-valued vector-fields throughout space and time, that is, the *E*- and *H*-fields, $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$, are real-valued vector functions of the spatial and temporal coordinates, $\boldsymbol{r}=(x,y,z)$ and t. It turns out, however, that complex-valued functions of $(\boldsymbol{r},t)$ in which $\boldsymbol{k}$, $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ are allowed to be "complex vectors" may be used to *mathematically represent* the *E*- and *H*-fields, so long as the real parts of these complex functions are recognized as corresponding to the actual fields. In this way, the physical fields always end up being real-valued, while their complex-valued *representations* simplify mathematical operations. Section 2 provides a detailed description of the complex vector algebra used throughout the chapter.

While a traditional, real-valued *k*-vector is used to represent a homogeneous plane-wave in free space or within a transparent medium, the use of a complex *k*-vector enables one to discuss inhomogeneous plane-waves such as evanescent waves in transparent media, exponentially decaying (or attenuating) plane-waves in absorptive media, and also exponentially growing (or amplifying) plane-waves within gain media. Moreover, the use of complex-valued field amplitudes $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ enable one to consider arbitrary states of polarization (i.e., linear, circular, elliptical) within the same mathematical formalism. A general discussion of complex-valued plane-waves and their properties is given in section 3.

Isotropic, homogeneous, linear media are characterized by the uniformity and isotropy of their electromagnetic properties, and by the fact that their dielectric susceptibility $\varepsilon_o\chi_e(\omega)$ relates the polarization density $\boldsymbol{P}(\boldsymbol{r},t)$ to the local *E*-field $\boldsymbol{E}(\boldsymbol{r},t)$, while their magnetic susceptibility

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$\mu_o\chi_m(\omega)$ relates the magnetization density $\boldsymbol{M}(\boldsymbol{r},t)$ to the local H-field $\boldsymbol{H}(\boldsymbol{r},t)$. Plane-wave solutions to Maxwell's equations in such media are discussed in section 4, where relationships among the frequency ω, the k-vector, the field amplitudes $(\boldsymbol{E}_o, \boldsymbol{H}_o)$, and material parameters $\varepsilon(\omega)=1+\chi_e(\omega)$ and $\mu(\omega)=1+\chi_m(\omega)$ are derived from Maxwell's macroscopic equations.

The rate of flow of electromagnetic energy (per unit area per unit time) in an optical or electromagnetic field is given by the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)$. A general expression for the time-averaged Poynting vector of a monochromatic plane-wave is derived in section 5. This result is applicable to all sorts of plane-waves, whether propagating in a transparent medium, exponentially decaying within an absorptive medium, exponentially growing inside a gain medium, or evanescent. Using the results of section 5, one can calculate the energy flux of any plane-wave residing in an isotropic, homogeneous, linear medium, and confirm the conservation of energy under various circumstances.

When an electromagnetic wave arrives at the interface between two (physically distinct) media, a part of the wave is reflected at the interface, while the remaining part enters from the first (incidence) medium into the second. Assuming the two media are isotropic, homogeneous, and linear – each specified by its dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$ – and also assuming that the two media occupy adjacent, semi-infinite, half-spaces separated at a flat interface, one can readily calculate the reflection and transmission coefficients for arbitrary plane-waves that arrive at the interface. These so-called Fresnel reflection and transmission coefficients are derived in section 6. The results of section 6 are completely general, and may be used in conjunction with transparent or absorptive media. They encompass reflection and transmission of arbitrarily-polarized plane-waves directed at the interface between two arbitrary media at an arbitrary angle of incidence. The media under consideration may be metals, plasmas, or dielectrics with positive or negative refractive indices. The results of section 6 are applicable to ordinary reflection and transmission, total internal reflection, and reflection from plasmas. We discuss several cases of practical interest in section 7.

1.1 The Electromagnetic Field

Review of complex vector algebra

This chapter describes the dependence of plane electromagnetic waves on space and time coordinates using a powerful complex notation. A complex number such as $c=a+\mathrm{i}b$ is specified by its real and imaginary parts a and b, respectively. The sum and difference of two complex numbers are defined as $c_1\pm c_2=(a_1\pm a_2)+\mathrm{i}(b_1\pm b_2)$. The product of two complex numbers is $c_1c_2=(a_1a_2-b_1b_2)+\mathrm{i}(a_1b_2+a_2b_1)$. Division is the inverse of multiplication, in the sense that if $c_3=c_1/c_2$, then $c_1=c_2c_3$. The complex-conjugate of $c=a+\mathrm{i}b$ is defined as $c^*=a-\mathrm{i}b$. The product $cc^*=a^2+b^2$ is always real and non-negative. It is often useful to write c_1/c_2 as $(c_1c_2^*)/(c_2c_2^*)$, which has a real-valued denominator; complex division is thus reduced to complex multiplication of c_1 and c_2^*, followed by normalization by the real-valued $c_2c_2^*$.

In *polar* representation, a complex number is written as $c=|c|\exp(\mathrm{i}\phi_c)$. The magnitude (or modulus) $|c|$ of the complex-number c is a non-negative real number; its phase ϕ_c is in the range $-\pi<\phi_c\le\pi$ (modulo 2π). The Euler identity $\exp(\mathrm{i}\theta)=\cos\theta+\mathrm{i}\sin\theta$ is generally used to convert c from one representation to the other, namely, $c=|c|\exp(\mathrm{i}\phi_c)=|c|\cos\phi_c+\mathrm{i}|c|\sin\phi_c=a+\mathrm{i}b$.

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A vector $\boldsymbol{V}$ in 3D-space is written in Cartesian coordinates as $\boldsymbol{V} = V_x\hat{\boldsymbol{x}} + V_y\hat{\boldsymbol{y}} + V_z\hat{\boldsymbol{z}}$. The sum and difference of two vectors are defined as $\boldsymbol{V}_1 \pm \boldsymbol{V}_2 = (V_{x1} \pm V_{x2})\hat{\boldsymbol{x}} + (V_{y1} \pm V_{y2})\hat{\boldsymbol{y}} + (V_{z1} \pm V_{z2})\hat{\boldsymbol{z}}$. As for vector multiplication, two types of products are defined in vector algebra. The rule for dot multiplying $\boldsymbol{V}_1 = V_{x1}\hat{\boldsymbol{x}} + V_{y1}\hat{\boldsymbol{y}} + V_{z1}\hat{\boldsymbol{z}}$ and $\boldsymbol{V}_2 = V_{x2}\hat{\boldsymbol{x}} + V_{y2}\hat{\boldsymbol{y}} + V_{z2}\hat{\boldsymbol{z}}$, denoted by $\boldsymbol{V}_1 \cdot \boldsymbol{V}_2$, is that the three terms of $\boldsymbol{V}_1$ are multiplied into the three terms of $\boldsymbol{V}_2$ using the following identities:

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = 1; \qquad \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}} = \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{z}} = \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{x}} = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{y}} = 0.$$

(Equation 1)

The other type of vector multiplication, denoted $\boldsymbol{V}_1 \times \boldsymbol{V}_2$ and referred to as cross-product, requires that the three terms of $\boldsymbol{V}_1$ be multiplied into the three terms of $\boldsymbol{V}_2$ using the alternative set of rules listed below:

$$\hat{\boldsymbol{x}}\times\hat{\boldsymbol{x}} = 0, \qquad \hat{\boldsymbol{y}}\times\hat{\boldsymbol{y}} = 0, \qquad \hat{\boldsymbol{z}}\times\hat{\boldsymbol{z}} = 0,$$

(Equation 2a)

$$\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}, \qquad \hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}, \qquad \hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}},$$

(Equation 2b)

$$\hat{\boldsymbol{y}}\times\hat{\boldsymbol{x}} = -\hat{\boldsymbol{z}}, \qquad \hat{\boldsymbol{z}}\times\hat{\boldsymbol{y}} = -\hat{\boldsymbol{x}}, \qquad \hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}} = -\hat{\boldsymbol{y}}.$$

(Equation 2c)

In the complex notation used in this chapter, we expand the domain of vector algebra by allowing the three components V_x, V_y, V_z of an arbitrary vector $\boldsymbol{V}$ to be complex numbers. Note that we *do not* change the rules for adding, subtracting, dot-multiplying, or cross-multiplying pairs of vectors; all we require is that the rules of complex-number addition and multiplication be followed when adding and multiplying the *components* V_{x1}, V_{y1}, V_{z1} of one vector with the *components* V_{x2}, V_{y2}, V_{z2} of another, while retaining the rules of vector multiplication for the unit vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ listed in **Equation.(1) and (2)**. A complex-valued vector may thus be written

$$\boldsymbol{V} = V_x\hat{\boldsymbol{x}} + V_y\hat{\boldsymbol{y}} + V_z\hat{\boldsymbol{z}} = (V_x' + \mathrm{i}V_x'')\hat{\boldsymbol{x}} + (V_y' + \mathrm{i}V_y'')\hat{\boldsymbol{y}} + (V_z' + \mathrm{i}V_z'')\hat{\boldsymbol{z}}.$$

(Equation 3a)

Combining the real parts of the various components of $\boldsymbol{V}$ into one real-valued vector $\boldsymbol{V}'$, and the imaginary parts into another (also real-valued) vector $\boldsymbol{V}''$, we rewrite **Equation.(3a)** as

$$\boldsymbol{V} = (V_x'\hat{\boldsymbol{x}} + V_y'\hat{\boldsymbol{y}} + V_z'\hat{\boldsymbol{z}}) + \mathrm{i}(V_x''\hat{\boldsymbol{x}} + V_y''\hat{\boldsymbol{y}} + V_z''\hat{\boldsymbol{z}}) = \boldsymbol{V}' + \mathrm{i}\boldsymbol{V}''.$$

(Equation 3b)

Each complex-valued vector $\boldsymbol{V}$ may therefore be considered a combination of two real-valued vectors $\boldsymbol{V}'$ and $\boldsymbol{V}''$ in accordance with **Equation.(3b).**

The reader should be warned that, although complex-numbers may, on occasion, be represented as 2D-vectors in the complex-plane, there must be no confusing such vectors with the 3D-vectors that are the subject of the present discussion. For our purposes, therefore, numbers in the complex-plane are scalars, *not* vectors. The only vectors we shall encounter are those in the conventional 3D-space; they are written as $\boldsymbol{V} = V_x\hat{\boldsymbol{x}} + V_y\hat{\boldsymbol{y}} + V_z\hat{\boldsymbol{z}}$, their components V_x, V_y, V_z being either real- or complex-valued. For example, a point $\boldsymbol{r}$ in 3D-space will be

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denoted in Cartesian coordinates as $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, a 3D-vector with real-valued components x, y, z. On the other hand, the propagation vector $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$ will be treated as a 3D-vector with complex components k_x, k_y, k_z.

In addition to the k-vector, we will use in this chapter complex vectors to denote the electric and magnetic field amplitudes $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$, corresponding to plane-waves propagating in isotropic, homogeneous, linear media. The advantage of using a complex k-vector is that it enables us to deal with propagating, evanescent, and attenuated plane-waves, all within the same formalism. Similarly, the advantage of identifying $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ with complex-vectors is that it enables a unified treatment of the linear, circular, and elliptical states of polarization.

Plane electromagnetic waves and their properties

By definition, a plane electromagnetic wave has E- and H-fields given by

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\{\boldsymbol{E}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\},$$

(Equation 4a)

$$\boldsymbol{H}(\boldsymbol{r},t) = \mathrm{Re}\{\boldsymbol{H}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}.$$

(Equation 4b)

In the most general case, $\boldsymbol{k}$, $\boldsymbol{E}_o$, and $\boldsymbol{H}_o$ are complex-valued vectors in ordinary 3D-space, which may be written

$$\boldsymbol{k} = \boldsymbol{k}' + \mathrm{i}\boldsymbol{k}''$$

(Equation 5a)

$$\boldsymbol{E}_o = \boldsymbol{E}_o' + \mathrm{i}\boldsymbol{E}_o''$$

(Equation 5b)

$$\boldsymbol{H}_o = \boldsymbol{H}_o' + \mathrm{i}\boldsymbol{H}_o''$$

(Equation 5c)

The temporal frequency ω associated with the time-rate-of-oscillation of the plane-wave can, in general, be a complex-valued scalar, e.g., $\omega = \omega' + \mathrm{i}\omega''$. However, for the purposes of this chapter, we set $\omega'' = 0$, thus assigning to ω a purely real-valued constant; the period of oscillations is then given by $T = 2\pi/\omega$.

The complex nature of the wave-vector $\boldsymbol{k}$ emphasizes its association with two real-valued 3D-vectors, $\boldsymbol{k}'$ and $\boldsymbol{k}''$, which, quite independently of each other, could have arbitrary magnitudes and be oriented in different directions in 3D-space. One may write $\boldsymbol{k}$ in terms of its Cartesian components $k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$, where $k_x = k_x' + \mathrm{i}k_x''$, $k_y = k_y' + \mathrm{i}k_y''$, $k_z = k_z' + \mathrm{i}k_z''$. It is obvious that the real parts of each Cartesian component of $\boldsymbol{k}$ define the (real-valued) vector $\boldsymbol{k}'$ as $k_x'\hat{\boldsymbol{x}} + k_y'\hat{\boldsymbol{y}} + k_z'\hat{\boldsymbol{z}}$, while the imaginary parts define the (real-valued) vector $\boldsymbol{k}''$ as $k_x''\hat{\boldsymbol{x}} + k_y''\hat{\boldsymbol{y}} + k_z''\hat{\boldsymbol{z}}$.

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The complex-valued amplitudes $\boldsymbol{E}_{\text{o}}$ and $\boldsymbol{H}_{\text{o}}$ are associated with the polarization state of the corresponding vector field. For instance, the E-field of the plane-wave has the following time- and space-dependences:

$$\boldsymbol{E}(\boldsymbol{r},t) = \text{Re}\{(\boldsymbol{E}_{\text{o}}'+\text{i}\boldsymbol{E}_{\text{o}}'')\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\exp[\text{i}(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)]\}$$

$$= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})[\boldsymbol{E}_{\text{o}}'\cos(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})+\boldsymbol{E}_{\text{o}}''\sin(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})].$$

(Equation 6)

The last expression on the right-hand side of **Equation.(6)** has an exponential term, $\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})$, which represents the decay of the field in the direction $\hat{\boldsymbol{k}}''$ of $\boldsymbol{k}''$, at a rate given by the magnitude k'' of $\boldsymbol{k}''$. It also has sinusoidal terms whose common argument $(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})$ signifies the propagation of the wave's phase-front in the direction $\hat{\boldsymbol{k}}'$ of $\boldsymbol{k}'$, with a wavelength $\lambda=2\pi/k'$ and a phase velocity $V_\phi=\omega/k'$. At a fixed point $\boldsymbol{r}$ in space, the E-field is aligned with $\pm\boldsymbol{E}_{\text{o}}'$ when $\cos(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})=\pm1$. A quarter of a period later, the E-field will be aligned with $\pm\boldsymbol{E}_{\text{o}}''$ when $\sin(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})=\pm1$. It is not difficult to show that, at any fixed location $\boldsymbol{r}$ in space, $\boldsymbol{E}(\boldsymbol{r},t)$ is an ordinary 3D-vector whose tip traces in time an ellipse defined by the (real-valued) vectors $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$, as depicted in **Figure.1.**

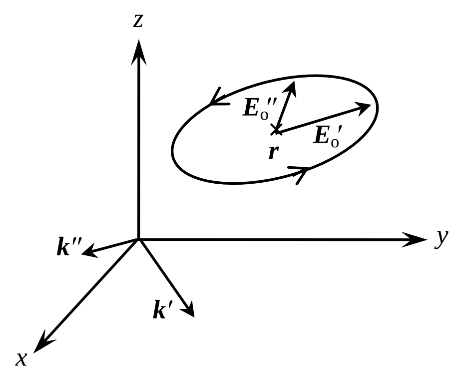

Figure.1. *The E-field $\boldsymbol{E}_o'\cos(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})+\boldsymbol{E}_o''\sin(\omega t-\boldsymbol{k}'\cdot\boldsymbol{r})$ at a fixed point $\boldsymbol{r}$ in space traverses an ellipse defined by the vectors $\boldsymbol{E}_o'$ and $\boldsymbol{E}_o''$. The period of each round trip around the ellipse is $T=2\pi/\omega$. The ellipse is centered on $\boldsymbol{r}$, and the end-points of $\boldsymbol{E}_o'$ and $\boldsymbol{E}_o''$ are located on the ellipse. In general, $\boldsymbol{E}_o'$ and $\boldsymbol{E}_o''$ are arbitrary, in the sense that no specific relationship needs to exist between their magnitudes, nor the angle between them has to have a particular value. Nevertheless, the ellipse is fully specified by $\boldsymbol{E}_o'$ and $\boldsymbol{E}_o''$. Also shown are the real and imaginary parts of the complex k-vector $\boldsymbol{k}'+\text{i}\boldsymbol{k}''$.*

Plane-waves in isotropic, homogeneous, linear media

Let us now assume that the plane-wave defined by **Equation.(4)** resides within a material medium which contains neither free charges nor free currents, that is, $\rho_{\text{free}}(\boldsymbol{r},t)=0$ and $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)=0$. Furthermore, the material medium, having complex-valued electric and magnetic susceptibilities $\varepsilon_{\text{o}}\chi_{\text{e}}(\omega)$ and $\mu_{\text{o}}\chi_{\text{m}}(\omega)$, is linearly polarizable and magnetizable such that

$$\boldsymbol{P}(\boldsymbol{r},t) = \text{Re}\{\varepsilon_{\text{o}}\chi_{\text{e}}(\omega)\boldsymbol{E}_{\text{o}}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\},$$

(Equation 7a)

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$$\boldsymbol{M}(\boldsymbol{r},t) = \mathrm{Re}\{\mu_o \chi_m(\omega)\boldsymbol{H}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

(Equation 7b)

The above equations lead directly to (complex-valued) electric permittivity $\varepsilon(\omega) = 1+\chi_e(\omega)$ and magnetic permeability $\mu(\omega) = 1+\chi_m(\omega)$ that relate the electric displacement $\boldsymbol{D}(\boldsymbol{r},t)$ to the E-field and the magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$ to the H-field, namely,

$$\boldsymbol{D}(\boldsymbol{r},t) = \mathrm{Re}\{\varepsilon_o \varepsilon(\omega)\boldsymbol{E}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\},$$

(Equation 8a)

$$\boldsymbol{B}(\boldsymbol{r},t) = \mathrm{Re}\{\mu_o \mu(\omega)\boldsymbol{H}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

(Equation 8b)

If we now substitute for the fields in Maxwell's macroscopic equations the plane-wave expressions of $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{H}(\boldsymbol{r},t)$, $\boldsymbol{D}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ given by **Equation.(4) and (8),** we find

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_{\mathrm{free}}(\boldsymbol{r},t) \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_o = 0;$$

(Equation 9a)

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_{\mathrm{free}}(\boldsymbol{r},t)+\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{H}_o = -\omega\varepsilon_o\varepsilon(\omega)\boldsymbol{E}_o;$$

(Equation 9b)

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{E}_o = \omega\mu_o\mu(\omega)\boldsymbol{H}_o;$$

(Equation 9c)

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{H}_o = 0.$$

(Equation 9d)

Substituting for $\boldsymbol{H}_o$ in **Equation.(9b)** from **Equation.(9c)**, and using the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ yields:

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_o) = (\boldsymbol{k}\cdot\boldsymbol{E}_o)\boldsymbol{k}-k^2\boldsymbol{E}_o = -\omega^2\mu_o\varepsilon_o\mu(\omega)\,\varepsilon(\omega)\boldsymbol{E}_o.$$

(Equation 10)

Using **Equation.(9a)** and the fact that the speed of light in vacuum is $c = 1/\sqrt{\mu_o\varepsilon_o}$, we obtain from **Equation.(10)** the following important relation between k^2, ω, c, the material permittivity $\varepsilon(\omega)$, and the material permeability $\mu(\omega)$:

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$$k^2 = (\omega/c)^2 \mu(\omega)\,\varepsilon(\omega).$$

(Equation 11)

Equation (11) is generally referred to as the *dispersion relation*, as it relates the (complex-valued) magnitude of the k-vector to the frequency ω and to the frequency-dependent material properties $\varepsilon(\omega)$, $\mu(\omega)$. Note that $k^2 = (\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'') = k'^2 - k''^2 + 2\mathrm{i}\,\boldsymbol{k}'\cdot\boldsymbol{k}''$ and, therefore,

$$k'^2 - k''^2 = (\omega/c)^2\,\mathrm{Re}[\mu(\omega)\,\varepsilon(\omega)],$$

(Equation 12a)

$$\boldsymbol{k}'\cdot\boldsymbol{k}'' = \tfrac{1}{2}(\omega/c)^2\,\mathrm{Im}[\mu(\omega)\,\varepsilon(\omega)].$$

(Equation 12b)

In general, at a given frequency ω, both $\mu(\omega)$ and $\varepsilon(\omega)$ can be at arbitrary locations within the complex plane. We remark on the following special cases, which arise when $\mu(\omega)$ and $\varepsilon(\omega)$ happen to be confined to specific regions of the complex plane.

1) $\mathrm{Im}[\varepsilon(\omega)] = 0$ and $\mathrm{Im}[\mu(\omega)] = 0$. The material medium exhibits neither energy gain nor loss. In accordance with Eq.(12b), the real and imaginary components of $\boldsymbol{k}$ are perpendicular to each other, that is, $\boldsymbol{k}' \perp \boldsymbol{k}''$. In this case, the real-valued product $\mu(\omega)\,\varepsilon(\omega)$ can be positive, negative, or zero. If k'' happens to be zero, the absence of the term $\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})$ in Eq.(6) would indicate that the plane-wave is neither amplified nor attenuated along its propagation path; the plane-wave is then referred to as *homogeneous*. Since $k'^2 = (\omega/c)^2\mu(\omega)\,\varepsilon(\omega)$ in this case, we must have $\mu(\omega)\,\varepsilon(\omega) \geq 0$. If, however, $k'' \neq 0$, the plane-wave will be *evanescent*, which is a special name given to *inhomogeneous* plane-waves that happen to have $\boldsymbol{k}' \perp \boldsymbol{k}''$. Here, according to Eq.(12a), $k' \geq k''$ if $\mu(\omega)\,\varepsilon(\omega) \geq 0$, otherwise $k' < k''$. An example of this type of situation arises for a plane-wave in vacuum, where $\mu(\omega) = \varepsilon(\omega) = 1$.

2) Media that exhibit no gain are referred to as passive; for these media $\mathrm{Im}[\varepsilon(\omega)] \geq 0$ and also $\mathrm{Im}[\mu(\omega)] \geq 0$. Under these circumstances, if $\mathrm{Im}[\mu(\omega)\,\varepsilon(\omega)] = 0$, the plane-wave will be either homogeneous or evanescent, otherwise it will be inhomogeneous, with $k' \neq 0$, $k'' \neq 0$, and the angle between $\boldsymbol{k}'$ and $\boldsymbol{k}''$ anything but 90º.

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Energy flux and the Poynting vector

The time-averaged Poynting vector for a plane-wave described by **Equation.(4-11)** may be written as follows:

$$<\boldsymbol{S}(\boldsymbol{r},t)> = ½\mathrm{Re}\{\boldsymbol{E}_o\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\times\boldsymbol{H}_o^*\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\} = ½\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}(\boldsymbol{E}_o\times\boldsymbol{H}_o^*)$$

$$= ½\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\{\boldsymbol{E}_o\times\{\boldsymbol{k}^*\times\boldsymbol{E}_o^*/[\omega\mu_o\mu^*(\omega)]\}\}$$

$$= (2\,\omega\mu_o\mu\,\mu^*)^{-1}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\{[(\boldsymbol{E}_o\cdot\boldsymbol{E}_o^*)\boldsymbol{k}^*-(\boldsymbol{E}_o\cdot\boldsymbol{k}^*)\boldsymbol{E}_o^*]\,\mu\}$$

$$= [2(\omega/c)Z_o\mu\,\mu^*]^{-1}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\{[(E_o'^2+E_o''^2)\boldsymbol{k}^*-\boldsymbol{k}^*\times(\boldsymbol{E}_o^*\times\boldsymbol{E}_o)-(\boldsymbol{k}\cdot\boldsymbol{E}_o)^*\boldsymbol{E}_o]\,\mu\}$$

$$= [2(\omega/c)Z_o\mu\,\mu^*]^{-1}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}[(E_o'^2+E_o''^2)\mu\boldsymbol{k}^*-2\mathrm{i}\,\mu\boldsymbol{k}^*\times(\boldsymbol{E}_o'\times\boldsymbol{E}_o'')].$$

In the above derivation we introduced the constant $Z_o=\sqrt{\mu_o/\varepsilon_o}$, the impedance of free space, and invoked Eq.(9a) when setting $\boldsymbol{k}\cdot\boldsymbol{E}_o=0$. The final expression can be further simplified to yield:

$$<\boldsymbol{S}(\boldsymbol{r},t)> = \frac{\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\left[(E_o'^2+E_o''^2)(\mu'\boldsymbol{k}'+\mu''\boldsymbol{k}'')-2(\mu'\boldsymbol{k}''-\mu''\boldsymbol{k}')\times(\boldsymbol{E}_o'\times\boldsymbol{E}_o'')\right]}{2Z_o(\omega/c)(\mu'^2+\mu''^2)}$$

(Equation 13)

Equation (13) is the general expression of time-averaged Poynting vector in isotropic, homogeneous, linear media whose electromagnetic properties are given by $\mu(\omega)$ and $\varepsilon(\omega)$ via **Equation.(7) and (8)**. The real and imaginary components $\boldsymbol{k}'$ and $\boldsymbol{k}''$ of the wave-vector satisfy **Equation.(12)**, and the *E*-field amplitude $\boldsymbol{E}_o$ satisfies Eq.(9a). The above expression is considerably simplified under special circumstances; for instance, in a transparent medium, where $\mathrm{Im}[\varepsilon(\omega)]=\mathrm{Im}[\mu(\omega)]=0$, for a *homogeneous* plane-wave (i.e., one having $\boldsymbol{k}''=0$) we have

$$<\boldsymbol{S}(\boldsymbol{r},t)> = ½(E_o'^2+E_o''^2)\boldsymbol{k}/[(\omega/c)Z_o\mu(\omega)] = ½(E_o'^2+E_o''^2)\hat{\boldsymbol{k}}/[Z_o\sqrt{\mu(\omega)/\varepsilon(\omega)}].$$

(Equation 14)

Note that the denominator of the expression on the right-hand side of Eq.(14), being the impedance of the transparent medium, is generally positive, even though the refractive index $n(\omega)=\sqrt{\mu(\omega)\,\varepsilon(\omega)}$ may be negative. Both components $\boldsymbol{E}_o'$ and $\boldsymbol{E}_o''$ of the *E*-field contribute to the energy flux, which now has the direction of the unit vector $\hat{\boldsymbol{k}}$ along the (real-valued) *k*-vector.

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1.2 Reflection and transmission of plane waves at the flat interface between adjacent media

In general, the k-vector of a plane-wave has three complex-valued components (k_x, k_y, k_z) along the axes of a Cartesian coordinate system. The constraint imposed by Eq.(11) allows one to determine, say, k_z from the knowledge of (k_x, k_y), but the latter two components need to be determined by some other means. Similarly, the three (complex) components of the E-field, (E_{xo}, E_{yo}, E_{zo}) are related to each other and to the k-vector through **Equation.(9a)**, but independent knowledge of (k_x, k_y, k_z) as well as (E_{xo}, E_{yo}) is needed before the value of E_{zo} could be extracted from Eq.(9a). Once $\boldsymbol{k}$ and $\boldsymbol{E}_o$ are determined, however, Eq.(9c) may be used to determine $\boldsymbol{H}_o$, which will then automatically satisfy **Equation.(9b) and (9d).**

In an isotropic system, one can always choose the coordinate axes such that $k_y = 0$. Thus the knowledge of (k_x, E_{xo}, E_{yo}) is all that is needed to fully specify a plane-wave. Now, consider the flat interface between two isotropic, homogeneous, linear media specified by $[\mu_a(\omega), \varepsilon_a(\omega)]$ on one side of the interface, and by $[\mu_b(\omega), \varepsilon_b(\omega)]$ on the other side, as in **Figure 2**. The interface is the xy-plane at $z = 0$, the incident plane-wave is identified by its minimal parameter set $(k_x^i, E_{xo}^i, E_{yo}^i)$, which is assumed to be known in advance, while the parameter sets of the reflected and transmitted plane-waves, $(k_x^r, E_{xo}^r, E_{yo}^r)$ and $(k_x^t, E_{xo}^t, E_{yo}^t)$, must be determined by matching the boundary conditions at $z = 0$.

The exponential factor $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ appearing in **Equation.(4)** reduces to $\exp[i(k_x x + k_y y - \omega t)]$ at the interface, where $z = 0$. The Maxwell boundary conditions at the interface require the continuity of E_x, E_y, D_z, H_x, H_y, and B_z at all interfacial points $(x, y, z = 0)$ for all times t. This is possible only if the following three conditions are satisfied:

1) $\omega^i = \omega^r = \omega^t$, namely, the reflected and transmitted waves have the same temporal frequency ω as the incident wave.

2) $k_x^i = k_x^r = k_x^t$, namely, the x-components of the reflected and transmitted k-vectors are identical to k_x^i, which is a parameter that is assumed to be known *a priori*. This is a generalized form of Snell's law of optics, which asserts that $n_a \sin\theta^i = n_a \sin\theta^r = n_b \sin\theta^t$ at the interface between two transparent media of refractive indices n_a and n_b, when the angles of incidence, reflection, and refraction are θ^i, θ^r, and θ^t, respectively.

3) $k_y^i = k_y^r = k_y^t$, namely, the y-components of the reflected and transmitted k-vectors are identical to k_y^i, which is also a parameter that is known *a priori*. In fact, in the preceding discussion, we set $k_y^i = 0$ and proceeded to assume that $k_y^r = k_y^t = 0$ as well. The latter assumption is now justified as a consequence of the constraints imposed by the boundary conditions on the reflected and transmitted waves at the interface.

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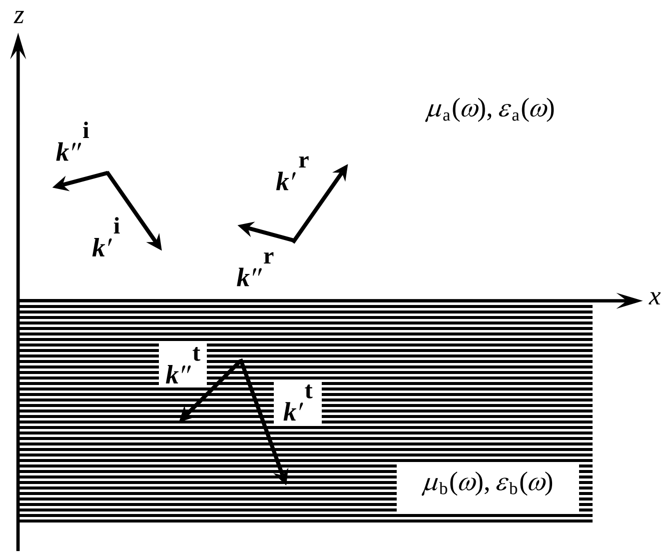

Figure. 2. *A plane-wave* $(\boldsymbol{k}^i, \omega^i, \boldsymbol{E}_o^i, \boldsymbol{H}_o^i)$ *is incident from above at the flat interface between two adjacent isotropic, homogeneous, linear media specified by their permittivity* $\varepsilon(\omega)$ *and permeability* $\mu(\omega)$. *The reflected and transmitted waves are identified by* $(\boldsymbol{k}^r, \omega^r, \boldsymbol{E}_o^r, \boldsymbol{H}_o^r)$ *and* $(\boldsymbol{k}^t, \omega^t, \boldsymbol{E}_o^t, \boldsymbol{H}_o^t)$, *respectively. The continuity of the tangential E- and H-fields at the interface requires that* $\omega^i = \omega^r = \omega^t$, $k_x^i = k_x^r = k_x^t$, *and* $k_y^i = k_y^r = k_y^t$. *The isotropy allows one to set* $k_y^i = k_y^r = k_y^t = 0$. *For each of the incident, reflected, and transmitted plane-waves, the knowledge of* (k_x, E_{xo}, E_{yo}) *is all that is needed to determine the remaining parameters via Maxwell's equations, as shown in Equation. (15). The boundary conditions further enable the reflected and transmitted* (E_{xo}, E_{yo}) *to be determined from a knowledge of the incident wave's properties.*

As it turns out, in consequence of the inherent redundancy of Maxwell's equations, the continuity of D_z *and* B_z *at the interface is automatically established once the tangential field components* E_x, E_y, H_x, *and* H_y *are made continuous at* $z = 0$. *The continuity requirement for* E_x, E_y, H_x, H_y *thus produces four linear equations in four unknowns – the unknown parameters at this point being* (E^r_{xo}, E^r_{yo}) *and* (E^t_{xo}, E^t_{yo}), *which are the only parameters that remain to be identified. Solving the field continuity equations at* $z = 0$ *is, therefore, the last step in determining the reflection and transmission coefficients for a monochromatic plane-wave incident at the flat interface between the two homogeneous, isotropic, linear media depicted in Figure. 2.*

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Assuming $k_y = 0$, Eq.(11) yields the following relation between k_x, ω/c, $\mu(\omega)$, and $\varepsilon(\omega)$:

$$k_z = \sqrt{(\omega/c)^2 \mu(\omega)\,\varepsilon(\omega) - k_x^2}.$$

(Equation 15a)

Also, from **Equation.(9a) and (9c)** we find

$$E_{zo} = -k_x E_{xo}/k_z,$$

(Equation 15b)

$$H_{xo} = -k_z E_{yo}/(\omega\mu_o\mu) \quad \rightarrow \quad (\omega/c)\,\mu Z_o H_{xo} = -k_z E_{yo},$$

(Equation 15c)

$$H_{yo} = (k_z E_{xo} - k_x E_{zo})/(\omega\mu_o\mu) \quad \rightarrow \quad k_z Z_o H_{yo} = (\omega/c)\,\varepsilon E_{xo},$$

(Equation 15d)

$$H_{zo} = k_x E_{yo}/(\omega\mu_o\mu) \quad \rightarrow \quad (\omega/c)\,\mu Z_o H_{zo} = k_x E_{yo}.$$

(Equation 15e)

Thus the boundary conditions can be separated into two independent groups, one for the (E_{xo}, H_{yo}) field components, commonly referred to as Transverse Magnetic (TM) or p-polarized waves, the other for the (E_{yo}, H_{xo}) field components, known as Transverse Electric (TE) or s-polarized waves. In the following paragraphs, we solve the two sets of equations separately and find the reflected and transmitted waves for p- and s-polarized plane-waves.

Case of TM or p-polarized incident plane-wave at a flat interface located at z = 0

The incident plane-wave is specified by $(k_x, k_z^{\mathrm{i}}, E_{xo}^{\mathrm{i}}, E_{zo}^{\mathrm{i}}, H_{yo}^{\mathrm{i}})$, as we assume that the incidence direction and the polarization state of the incident beam are chosen such that $k_y = 0$ and $E_{yo}^{\mathrm{i}} = 0$. The independent incidence parameters are thus k_x and E_{xo}^{i}, as k_z^{i} is related to k_x through Eq.(15a), while E_{zo}^{i} and H_{yo}^{i} are related to k_x and E_{xo}^{i} through **Equation.(15b) and (15d)**, respectively. The only independent parameter of the reflected beam is E_{xo}^{r}, as k_x is the same for all three beams (incident, reflected, transmitted), and the remaining parameters are related through **Equation.(15)**. Similarly, the only independent parameter of the transmitted beam is E_{xo}^{t}. The two unknowns in this problem, therefore, are E_{xo}^{r}, E_{xo}^{t}, which can be determined through the continuity equations for E_x and H_y at the interface, namely,

$$E_{xo}^{\mathrm{i}} + E_{xo}^{\mathrm{r}} = E_{xo}^{\mathrm{t}},$$

(Equation16a)

$$H_{yo}^{\mathrm{i}} + H_{yo}^{\mathrm{r}} = H_{yo}^{\mathrm{t}} \quad \rightarrow \quad \frac{\varepsilon_{\mathrm{a}} E_{xo}^{\mathrm{i}}}{\sqrt{(\omega/c)^2 \mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - k_x^2}} - \frac{\varepsilon_{\mathrm{a}} E_{xo}^{\mathrm{r}}}{\sqrt{(\omega/c)^2 \mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - k_x^2}} = \frac{\varepsilon_{\mathrm{b}} E_{xo}^{\mathrm{t}}}{\sqrt{(\omega/c)^2 \mu_{\mathrm{b}}\varepsilon_{\mathrm{b}} - k_x^2}}.$$

(Equation 16b)

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Note in **Equation.(16b)** that we have chosen opposite signs for $k_z^{\rm r}$ and $k_z^{\rm i}$, since the propagation direction of the reflected beam along the *z*-axis must be the reverse of the incident beam. The above equations may now be solved to yield the Fresnel reflection and transmission coefficients $\rho_{\rm p}$ and $\tau_{\rm p}$ for p-polarized plane-waves as follows:

$$\rho_{\rm p} = E^{\rm r}_{x{\rm o}}/E^{\rm i}_{x{\rm o}} = \frac{\varepsilon_{\rm a}\sqrt{\mu_{\rm b}\varepsilon_{\rm b}-(ck_x/\omega)^2} - \varepsilon_{\rm b}\sqrt{\mu_{\rm a}\varepsilon_{\rm a}-(ck_x/\omega)^2}}{\varepsilon_{\rm a}\sqrt{\mu_{\rm b}\varepsilon_{\rm b}-(ck_x/\omega)^2} + \varepsilon_{\rm b}\sqrt{\mu_{\rm a}\varepsilon_{\rm a}-(ck_x/\omega)^2}}, \qquad \textit{(Equation 17a)}$$

$$\tau_{\rm p} = E^{\rm t}_{x{\rm o}}/E^{\rm i}_{x{\rm o}} = \frac{2\varepsilon_{\rm a}\sqrt{\mu_{\rm b}\varepsilon_{\rm b}-(ck_x/\omega)^2}}{\varepsilon_{\rm a}\sqrt{\mu_{\rm b}\varepsilon_{\rm b}-(ck_x/\omega)^2} + \varepsilon_{\rm b}\sqrt{\mu_{\rm a}\varepsilon_{\rm a}-(ck_x/\omega)^2}}. \qquad \textit{(Equation 17b)}$$

Equations (17) are the most general form of reflection and transmission coefficients for p-polarized light at the interface between two isotropic, homogeneous, linear media specified in terms of their dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$. These results can be readily simplified in the following special cases:

1) The case of normal incidence is obtained by setting $k_x = 0$.

2) For incidence from the free-space, set $\varepsilon_{\rm a} = \mu_{\rm a} = 1.0$.

3) At optical frequencies, typical materials have no magnetic response; the corresponding $\rho_{\rm p}$ and $\tau_{\rm p}$ are thus obtained by setting $\mu_{\rm a} = \mu_{\rm b} = 1.0$.

4) Total internal reflection from a transparent medium whose $(\varepsilon_{\rm a}, \mu_{\rm a})$ are both real and positive, onto another transparent medium with real and positive $(\varepsilon_{\rm b}, \mu_{\rm b})$, is obtained when $\sqrt{\mu_{\rm b}\varepsilon_{\rm b}} < ck_x/\omega < \sqrt{\mu_{\rm a}\varepsilon_{\rm a}}$. Here $|\rho_{\rm p}| = 1$, while $\tau_{\rm p}$ specifies the amplitude of the evanescent wave in the transmission medium.

Case of TE or s-polarized incident plane-wave at a flat interface located at z = 0

The incident plane-wave is specified by $(k_x, k_z^{\rm i}, E^{\rm i}_{y{\rm o}}, H^{\rm i}_{x{\rm o}}, H^{\rm i}_{z{\rm o}})$, as we assume that the incidence direction and the polarization state of the incident beam are chosen such that $k_y = 0$ and $E^{\rm i}_{x{\rm o}} = E^{\rm i}_{z{\rm o}} = 0$. The independent incidence parameters are thus k_x and $E^{\rm i}_{y{\rm o}}$, as $k_z^{\rm i}$ is related to k_x through Eq.(15a), while $H^{\rm i}_{x{\rm o}}$ and $H^{\rm i}_{z{\rm o}}$ are related to k_x and $E^{\rm i}_{y{\rm o}}$ through **Equation.(15c) and (15e)**, respectively. The only independent parameter of the reflected beam is $E^{\rm r}_{y{\rm o}}$, as k_x is the same for all three beams (incident, reflected, transmitted), and the remaining parameters are related through **Equation.(15)**. Similarly, the only independent parameter of the transmitted beam is $E^{\rm t}_{y{\rm o}}$. The two unknowns in this problem, therefore, are $E^{\rm r}_{y{\rm o}}$, $E^{\rm t}_{y{\rm o}}$, which can be determined through the continuity equations for E_y and H_x at the interface, namely,

$$E^{\rm i}_{y{\rm o}} + E^{\rm r}_{y{\rm o}} = E^{\rm t}_{y{\rm o}}, \qquad \textit{(Equation 18a)}$$

$$H^{\rm i}_{x{\rm o}} + H^{\rm r}_{x{\rm o}} = H^{\rm t}_{x{\rm o}}$$

$$\rightarrow \quad \mu_{\rm a}^{-1}\sqrt{(\omega/c)^2\mu_{\rm a}\varepsilon_{\rm a} - k_x^2}\, E^{\rm i}_{y{\rm o}} - \mu_{\rm a}^{-1}\sqrt{(\omega/c)^2\mu_{\rm a}\varepsilon_{\rm a} - k_x^2}\, E^{\rm r}_{y{\rm o}} = \mu_{\rm b}^{-1}\sqrt{(\omega/c)^2\mu_{\rm b}\varepsilon_{\rm b} - k_x^2}\, E^{\rm t}_{y{\rm o}}. \qquad \textit{(Equation 18b)}$$

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Note in **Equation.(18b)** that we have chosen opposite signs for k_z^{r} and k_z^{i}, since the propagation direction of the reflected beam along the z-axis must be the reverse of the incident beam. The above equations may now be solved to yield the Fresnel reflection and transmission coefficients ρ_{s} and τ_{s} for s-polarized plane-waves as follows:

$$\rho_{\mathrm{s}} = E_{yo}^{\mathrm{r}}/E_{yo}^{\mathrm{i}} = \frac{\mu_{\mathrm{b}}\sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2} - \mu_{\mathrm{a}}\sqrt{\mu_{\mathrm{b}}\varepsilon_{\mathrm{b}} - (ck_x/\omega)^2}}{\mu_{\mathrm{b}}\sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2} + \mu_{\mathrm{a}}\sqrt{\mu_{\mathrm{b}}\varepsilon_{\mathrm{b}} - (ck_x/\omega)^2}}, \qquad \textit{(Equation 19a)}$$

$$\tau_{\mathrm{s}} = E_{yo}^{\mathrm{t}}/E_{yo}^{\mathrm{i}} = \frac{2\mu_{\mathrm{b}}\sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2}}{\mu_{\mathrm{b}}\sqrt{\mu_{\mathrm{a}}\varepsilon_{\mathrm{a}} - (ck_x/\omega)^2} + \mu_{\mathrm{a}}\sqrt{\mu_{\mathrm{b}}\varepsilon_{\mathrm{b}} - (ck_x/\omega)^2}}. \qquad \textit{(Equation 19b)}$$

Equations (19) are the most general form of reflection and transmission coefficients for s-polarized light at the interface between two isotropic, homogeneous, linear media specified in terms of their dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$. As before, these results can be readily simplified in special cases.

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1.3 Fresnel reflection and transmission coefficients

We now discuss the application of the results obtained in the preceding sections to a few cases that are of special importance in practice. Before embarking on a discussion of specific examples, we make the following observations.

i) In vacuum, $\varepsilon(\omega)=\mu(\omega)=1$.

ii) In ordinary materials at optical frequencies and beyond (i.e., $\omega>10^{14}$ rad/sec) one can set $\mu(\omega)=1$, as these media do not exhibit any significant magnetic response to the passage of light waves.

iii) Homogeneous plane-waves, which exist in free-space and within transparent materials, have real-valued k-vectors. In accordance with Eq. (11), the magnitude k of such plane-waves is given by $k=(\omega/c)n(\omega)$, where $n(\omega)=\sqrt{\mu(\omega)\,\varepsilon(\omega)}$ is positive when both $\mu(\omega)$ and $\varepsilon(\omega)$ are real and positive. In the so-called negative-index media, where $\mu(\omega)$ and $\varepsilon(\omega)$ are both real but negative at the frequency of interest, the refractive index $n(\omega)$ is real-valued and negative.

iv) With reference to the geometry of **Figure 2**, the real-valued k-vector of a *homogeneous* plane-wave is confined to the xz-plane. Denoting by θ^{i} the incidence angle, i.e., the angle between the normal to the interface (the z-axis) and the direction of incidence, we will have, for a homogeneous incident plane-wave, $k_x=k\sin\theta^{\mathrm{i}}=(\omega/c)n(\omega)\sin\theta^{\mathrm{i}}$. By definition, at normal incidence $\theta^{\mathrm{i}}=0$, while at grazing incidence $\theta^{\mathrm{i}}=90^{\circ}$. Incidentally, since the incident and reflected plane-waves are in the same medium, the equality of k_x for the two waves guarantees that $\theta^{\mathrm{r}}=\theta^{\mathrm{i}}$, namely, that the incidence and reflectance angles are always the same.

v) In the literature, there are sometimes references to *reflectivity* or *reflectance* of an interface. By definition, reflectivity (or reflectance) for p- and s-polarized plane-waves are $R_{\mathrm{p}}=|\rho_{\mathrm{p}}|^2$, and $R_{\mathrm{s}}=|\rho_{\mathrm{s}}|^2$, respectively. Since the incident and reflected plane-waves are in the same medium, and also since $\theta^{\mathrm{r}}=\theta^{\mathrm{i}}$, one can consider R_{p} (or R_{s}) as the ratio of reflected to incident optical energy flux per unit area per unit time, i.e., the ratio of the corresponding Poynting vectors.

Special Case 1: normal incidence.

At normal incidence $k_x=0$, in which case p- and s-polarized plane-waves become indistinguishable. Under these circumstances, **Equation. (17) and (19)** yield identical results for the Fresnel reflection and transmission coefficients, as follows:

$$\rho=\frac{\sqrt{\varepsilon_{\mathrm{a}}/\mu_{\mathrm{a}}}-\sqrt{\varepsilon_{\mathrm{b}}/\mu_{\mathrm{b}}}}{\sqrt{\varepsilon_{\mathrm{a}}/\mu_{\mathrm{a}}}+\sqrt{\varepsilon_{\mathrm{b}}/\mu_{\mathrm{b}}}}, \qquad \textit{(Equation 20a)}$$

$$\tau=\frac{2\sqrt{\varepsilon_{\mathrm{a}}/\mu_{\mathrm{a}}}}{\sqrt{\varepsilon_{\mathrm{a}}/\mu_{\mathrm{a}}}+\sqrt{\varepsilon_{\mathrm{b}}/\mu_{\mathrm{b}}}}. \qquad \textit{(Equation 20b)}$$

When the two media are impedance-matched, that is, when $\sqrt{\varepsilon_{\mathrm{a}}/\mu_{\mathrm{a}}}=\sqrt{\varepsilon_{\mathrm{b}}/\mu_{\mathrm{b}}}$, reflectivity drops to zero, in which case the entire incident plane-wave will be transmitted into the second medium.

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Special Case 2: Brewster's angle.

At optical frequencies where, for conventional media, $\mu_a(\omega)=\mu_b(\omega)=1$, consider the interface between two *transparent* media of refractive indices $n_a(\omega)=\sqrt{\varepsilon_a(\omega)}$ and $n_b(\omega)=\sqrt{\varepsilon_b(\omega)}$. Let the incident plane-wave be homogeneous, in which case $k_x=(\omega/c)n_a(\omega)\sin\theta^i$. Setting $\rho_p=0$ in Eq.(17a) yields $\sin^2\theta^i=\varepsilon_b/(\varepsilon_a+\varepsilon_b)$, or, equivalently, $\tan\theta^i=n_b/n_a$. Therefore, for p-polarized light, there is always an incidence angle, known as the Brewster angle $\theta_B=\tan^{-1}(n_b/n_a)$, for which the Fresnel reflection coefficient vanishes. Inspection of Eq.(19a) shows that no such angle exists for s-polarized light.

With the aid of Snell's law, $n_a\sin\theta^i=n_b\sin\theta^t$, it is now easy to show that, when $\theta^i=\theta_B$, the refraction angle will be $\theta^t=\theta_B'=\tan^{-1}(n_a/n_b)$. In other words, $\theta_B+\theta_B'=90°$, namely, at Brewster's incidence, the incident and refracted *k*-vectors are perpendicular to each other.

Special Case 3: total internal reflection

Consider once again the interface between two *transparent* media of refractive indices $n_a(\omega)=\sqrt{\varepsilon_a(\omega)}$ and $n_b(\omega)=\sqrt{\varepsilon_b(\omega)}$ at optical frequencies where $\mu_a(\omega)=\mu_b(\omega)=1$. At a given frequency ω, assume $n_a>n_b$, and consider a *homogeneous* incident plane-wave having $k_x=(\omega/c)n_a(\omega)\sin\theta^i$. Defining the critical angle of total internal reflection (TIR) as $\theta_c=\sin^{-1}(n_b/n_a)$, we find from **Equation.(17a) and (19a)** that, when $\theta^i>\theta_c$, one may write

$$\rho_p=\frac{i\sqrt{\sin^2\theta^i-\sin^2\theta_c}-\sin^2\theta_c\cos\theta^i}{i\sqrt{\sin^2\theta^i-\sin^2\theta_c}+\sin^2\theta_c\cos\theta^i}, \qquad \textit{(Equation 21a)}$$

$$\rho_s=\frac{\cos\theta^i-i\sqrt{\sin^2\theta^i-\sin^2\theta_c}}{\cos\theta^i+i\sqrt{\sin^2\theta^i-\sin^2\theta_c}}. \qquad \textit{(Equation 21b)}$$

In each of the above expressions, the numerator and the denominator are complex conjugates of each other; therefore, $|\rho_p|=1$ and $|\rho_s|=1$, meaning that in both cases, when $\theta^i>\theta_c=\sin^{-1}(n_b/n_a)$, the incident light is totally reflected at the interface. The only difference between ρ_p and ρ_s is in the phase of the reflected light, which may be readily calculated from **Equation.(21)**.

When total internal reflection occurs at the interface between two transparent dielectrics, the transmitted field, which resides in the dielectric medium below the interface, becomes evanescent. From the equations given in the preceding sections, one can calculate the properties of this evanescent field and find out, for example, that its time-averaged component of the Poynting vector perpendicular to the interface is always equal to zero, namely, $<S_z(\boldsymbol{r},t)>=0$.

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1.4 Reflection and transmission of guided modes in optical fibers

The effective refractive index of a waveguide, such as an optical fiber, quantifies the phase delay per unit length of the waveguide relative to the corresponding delay for propagation in vacuum. The refractive index n of an isotropic, homogeneous, and transparent medium represents the increase in the wavenumber k, which is the phase change per unit propagation distance within the medium: the wavenumber is n times higher than it would be in vacuum, i.e., $k = nk_o = 2\pi n/\lambda$. The effective refractive index n_{eff} has the analogous meaning for light propagation in a waveguide; the propagation constant is the effective index times the vacuum wavenumber: $\beta = 2\pi n_{eff}/\lambda$. The effective index of a multimode waveguide thus depends not only on the wavelength λ but also on the mode in which the light propagates. In general n_{eff} may be complex-valued, with its imaginary part signifying gain or loss.

Modal propagation in a waveguide

Consider the slab waveguide depicted in **Figure.1**. The guiding layer has thickness d and refractive index n_g. The substrate and the cladding layer, having refractive indices n_s and n_c, respectively, may be assumed to be infinitely thick. Within the guiding layer a pair of plane-waves propagate at an angle θ relative to the surface normal; θ is greater than the critical angle of total internal reflection at both interfaces, that is, $n_g \sin\theta > \max(n_s, n_c)$. The two plane-waves thus have the following complex amplitudes:

$$E_{\pm}(x, z) = |E_o| \exp(\pm i\phi_o) \exp[i(2\pi n_g/\lambda_o)(\pm x \cos\theta + z \sin\theta)].$$

(Equation 22)

Here the plus sign refers to the up-going beam, the minus sign to the down-going beam, ϕ_o defines the relative phase between the two plane-waves, and $\lambda_o = c/f_o$ is the vacuum wavelength.

At the interface with the cladding, where $x = \frac{1}{2}d$, the down-going beam must have the same amplitude as the up-going beam, but its phase must be incremented by the phase of the Fresnel reflection coefficient at this interface. The Fresnel coefficient, depending on whether the beam is s- or p-polarized, is $r_p = \exp(i\phi_p)$ or $r_s = \exp(i\phi_s)$, where

$$\phi_p^{(clad)} = +2\tan^{-1}[(n_c^2\cos\theta)/(n_g\sqrt{n_g^2\sin^2\theta - n_c^2})],$$

(Equation 23)

$$\phi_s^{(clad)} = -2\tan^{-1}[\sqrt{n_g^2\sin^2\theta - n_c^2}\,/(n_g\cos\theta)].$$

(Equation 24)

Therefore, at the cladding interface, one must have

$$\phi_o + (2\pi n_g/\lambda_o)(\tfrac{1}{2}d\cos\theta + z\sin\theta) + \phi_{p,s}^{(clad)} = -\phi_o + (2\pi n_g/\lambda_o)(-\tfrac{1}{2}d\cos\theta + z\sin\theta),$$

(Equation 25)

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which leads to

$$2\phi_o + 2\pi n_g(d/\lambda_o)\cos\theta + \phi_{p,s}^{(clad)} = 0.$$

(Equation 26)

A similar relation must hold at the substrate interface ($x = -½d$), where the down-going beam is incident and the up-going beam is reflected. Therefore,

$$-2\phi_o + 2\pi n_g(d/\lambda_o)\cos\theta + \phi_{p,s}^{(sub)} = 0.$$

(Equation 27)

Equations (5) and (6) can be satisfied simultaneously if and only if an integer m exists such that

$$4\pi n_g(d/\lambda_o)\cos\theta + \phi_{p,s}^{(clad)} + \phi_{p,s}^{(sub)} = 2m\pi.$$

(Equation 28)

If the guiding layer's thickness d is sufficiently small, Eq. (7) will have only one solution (i.e., one acceptable value of θ) for s-light, and perhaps another solution for p-light. The guide is then said to be single-mode. Larger values of d lead to more solutions, which correspond to higher-order modes. (Note: $\theta = 90°$ is always an acceptable solution; however, ϕ_o in this case turns out to be 0° for p-light and 90° for s-light. Both of these solutions result in the up-going and down-going plane-waves coming into alignment with equal but opposite amplitudes, thereby canceling each other out. The solution corresponding to $\theta = 90°$, therefore, does *not* lead to a viable mode.) For a viable mode, denoting the solution of Eq. (7) by θ_m, and with reference to Eq. (1), the E-field amplitude within the guiding layer will be

$$E(x, z) = E_+(x, z) + E_-(x, z)$$
$$= 2|E_o|\cos[(2\pi n_g\cos\theta_m/\lambda_o)x + \phi_o]\exp[i(2\pi n_g\sin\theta_m/\lambda_o)z].$$

(Equation 29)

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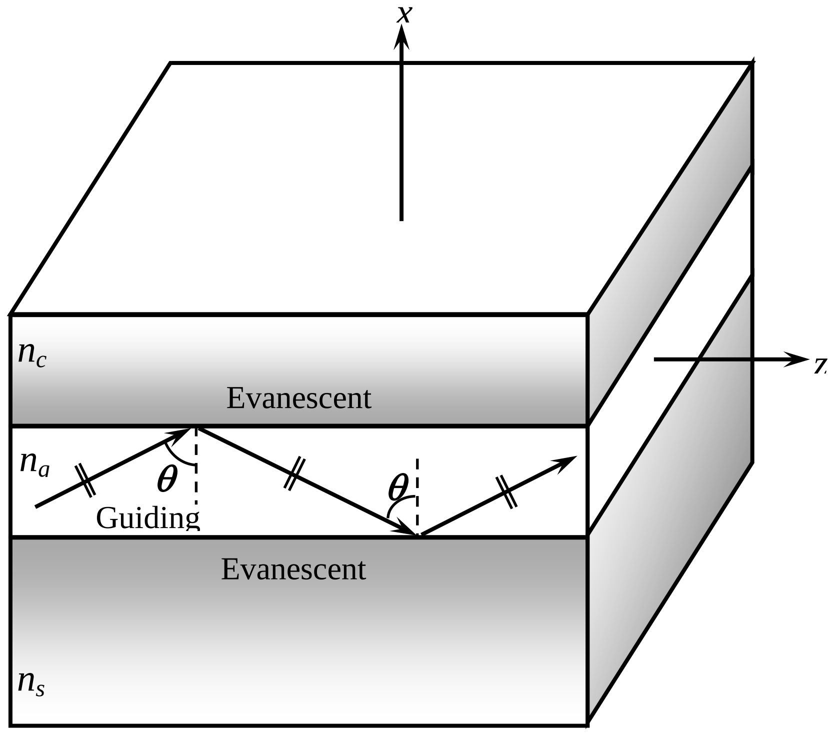

Figure 1. *Slab waveguide consisting of a guiding layer of thickness d and refractive index n_g, sandwiched between a substrate of index n_s and a cladding layer of index n_c. Within the guiding layer, a pair of plane-waves propagate at an angle θ relative to the surface normal.*

The cross-sectional profile of the mode along the x-axis is thus determined by the cosine function on the right-hand side of Eq. (8) (and also by the evanescent fields within the cladding and the substrate). The exponential term in Eq. (8) is the propagation phase-factor, from which one can identify an effective refractive index $n_{eff} = n_g \sin\theta_m$ for the given mode. Considering that, in general, both n_g and the solution θ_m of Eq. (7) are functions of the frequency f, the dispersive properties of the waveguide are seen to arise from the frequency dependence of n_{eff}.

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