Module 8 – Rate Equations and Multimode Operation of Semicon-ductor Lasers

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Introduction

One of the unique attributes of semiconductor lasers is their spectral output, which consists of multiple, widely spaced lines (**Figure 8.1**). In this module we will investigate the source of multimode operation in Fabry-Perot-type, edge-emitting, semiconductor lasers.

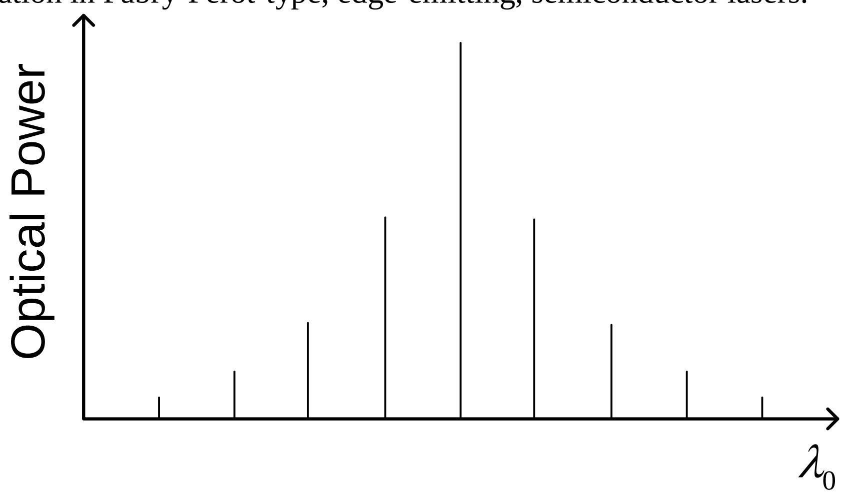

Figure 8.1. *Optical output from a Fabry-Perot-type semiconductor laser.*

In the first section of the module, we will assemble the rate equations that are needed for a quantitative exploration of the laser. In the second section, we will produce analytical solutions of simplified versions of the rate equations that suggest semiconductor lasers should operate in a single longitudinal mode. The argument for single mode operation can be summarized as follows: “The gain curve is homogeneously broadened, so all modes see the same gain curve. Only one of the modes is at a wavelength where the gain is sufficient to balance loss and initiate

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lasing.” In the last section, we will numerically solve more accurate versions of the rate equations and find that we indeed expect multimode operation. The rebuttal argument will be: “Yes the gain curve is homogeneously broadened, but the presence of strong amplified spontaneous emission means that there can be a significant number of photons in multiple modes that have wavelengths near the peak of the gain curve.”

8.1 Rate Equations

To start with we will assume there is only one optical mode in the laser. We can think of this approach in two ways, first as an approximation to the true multimode operation of the laser, which can be used to understand the source of multimode lasing. Second, we can think of single mode operation as an accurate depiction of the operation of a single mode device such as a distributed feedback laser.

8.1.1 Current Injection

Most semiconductor lasers are electrically pumped. An electrical current supply is connected to two electrical leads that are electrically connected to the p and n-type layers of the laser. Electrons from the n-type layer are injected into the conduction band of the active region of the laser while holes from the p-type layer are injected in the valence band of the active region of the laser as shown in **Figure 8.2**. The letter n will be used to specify the density of electrons in the active layer. The quantity n has units of number per unit volume – usually a cubic centimeter. The letter p will be used to specify the density of holes in the active layer. It is typically assumed that coulombic restoring forces keep the active region electrically neutral and that the n = p.

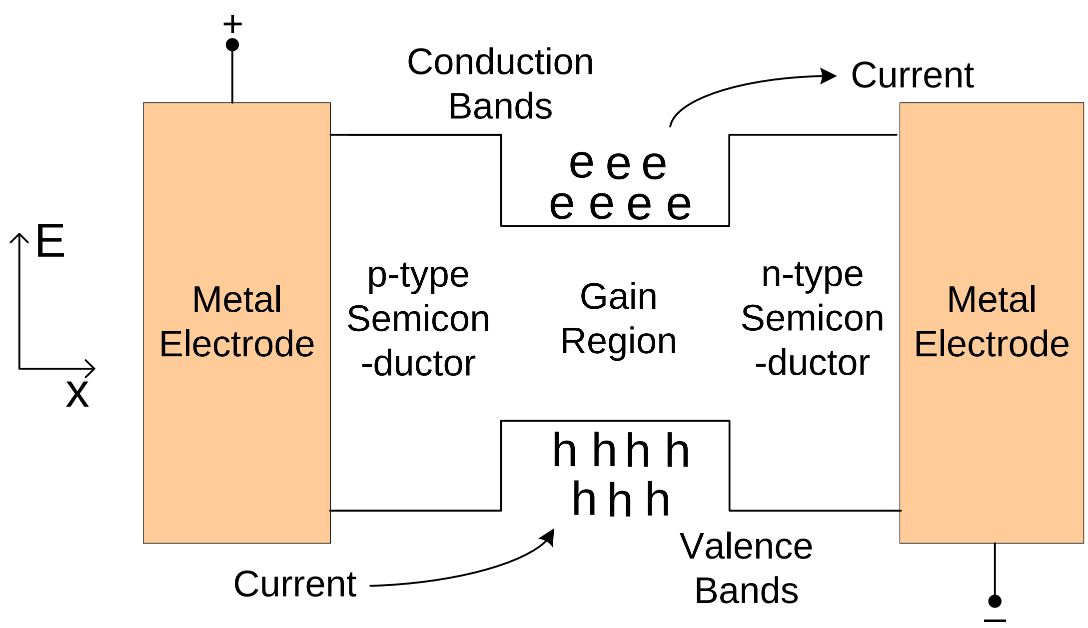

Figure 8.2.. *Electrical pumping of a semiconductor laser.*

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We will denote the total number of photons in the optical mode by the symbol ϕ. The photons circulate in the laser with a velocity $v_G = c/n_G$, where v_g is the group velocity for the photons in the laser medium and n_G is the group index for the optical mode (**Figure 8.3**). While many of the photons may overlap the active volume V of the laser, a certain fraction of the photons will travel in regions outside of the active volume.

Because we assume n = p, it will be sufficient to establish rate equations for n alone. The starting point is simply a statement that the rate of change of n with time is equal to the generation rate minus the electron-hole recombination rate:

$$\frac{dn}{dt} = G_{gen} - R_{rec}.$$

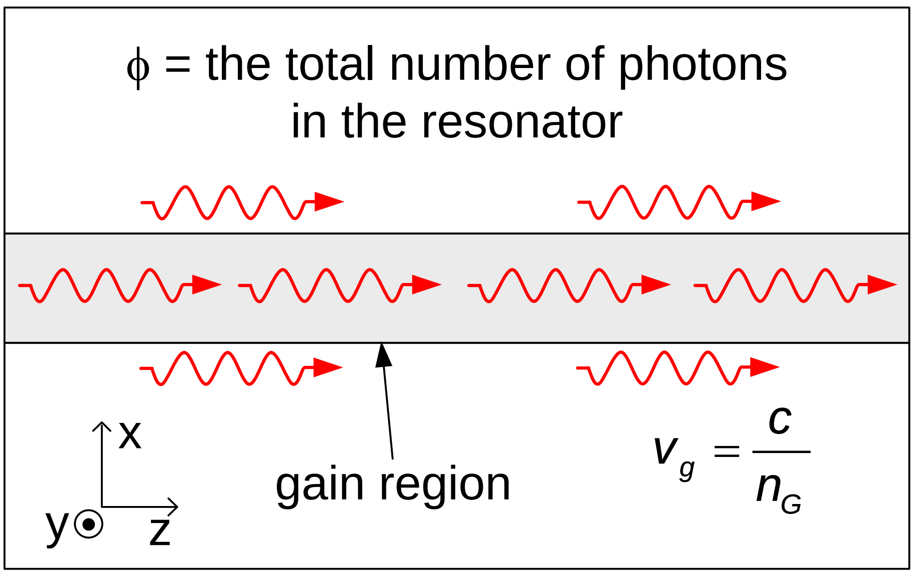

Figure 8.3. *A quantity ϕ photons circulate in the laser resonator with a velocity c/n_G.*

The generation rate is given by the current supplied to the diode divided by the electronic charge, and also divided by the volume of the active region to give units of number per unit volume per unit time.

$$G_{gen} = \frac{I}{qV}$$

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8.1.2 Electron-Hole Recombination

The electron hole recombination rate is the sum of rates from four contributions.

$$R_{rec} = \underbrace{R_{srh}}_{\substack{\text{Shockley-Read-Hall}\\ \text{Recombination}}} + \underbrace{R_{se}}_{\substack{\text{Spontaneous}\\ \text{Emission}}} + \underbrace{R_{Auger}}_{\substack{\text{Auger}\\ \text{Recombination}}} + \underbrace{R_{st}}_{\substack{\text{Stimulated}\\ \text{Emission}}}$$

8.1.3 Shockley-Read-Hall Recombination

Shockley-Read-Hall (SRH) recombination is called an “extrinsic” process because it requires the presence of impurity states in the energy gap that would be absent in an ideal crystal. SRH recombination proceeds in the manner illustrated in **Figure 8.4**. A charged particle, say an electron, encounters a trap state in the middle of the forbidden gap of a semiconductor. There the electron waits for a hole to pass nearby. When the hole encounters the electron, they recombine. SRH recombination may produce infrared photons, but it is common to refer to this as a non-radiative process because any photons produced are outside of the region of interest.

It often turns out that the SRH recombination rate is proportional to density of electrons (or holes since n=p). Thus we can write

$$R_{srh} = A_{nr} n .$$

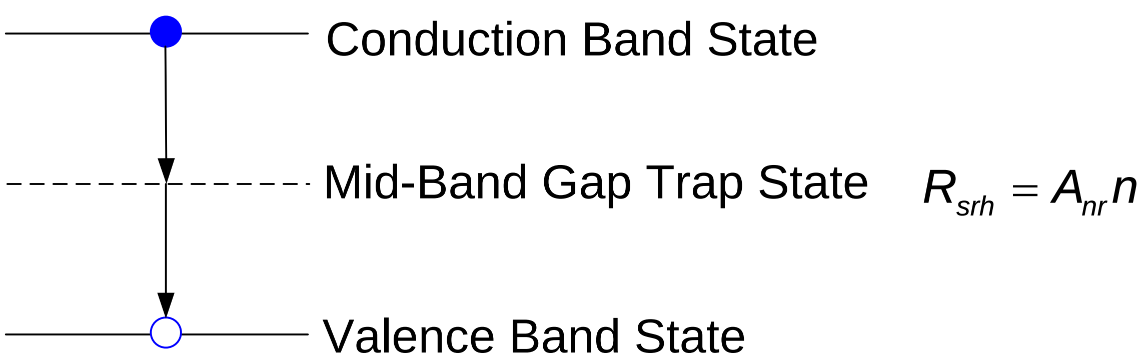

Figure 8.4. *Shockley-Read-Hall recombination of an electron with a hole.*

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8.1.4 Spontaneous Emission

Spontaneous emission (**Figure 8.5**) is a "two body" process (an electron and a hole) with a rate

$$R_{se} = Bnp = Bn^2.$$

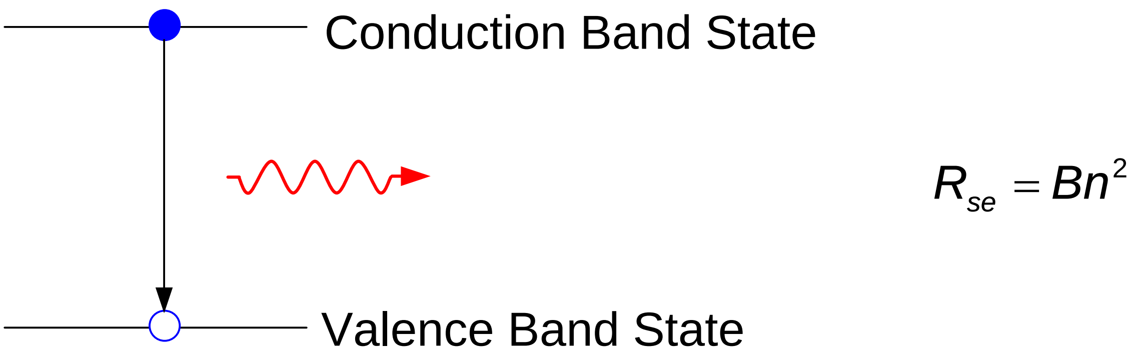

Figure 8.5. *Electron-hole recombination by spontaneous emission.*

8.1.5 Auger Recombination

Auger recombination is an "intrinsic" recombination mechanism that can occur in an ideal crystal without imperfections. There are several types of Auger recombination. The process illustrated in **Figure 8.6** is called a CCCH Auger process because it involves three electron states in a conduction band and a hole in a valence band.

In the CCCH process, an electron in the conduction band recombines with a hole. However, instead of producing a photon, the energy from the recombination goes to promote a second electron in the conduction band to an energy that is higher in the band. The excited electron makes its way back to the bottom of the conduction band, emitting phonons (heat) on its way.

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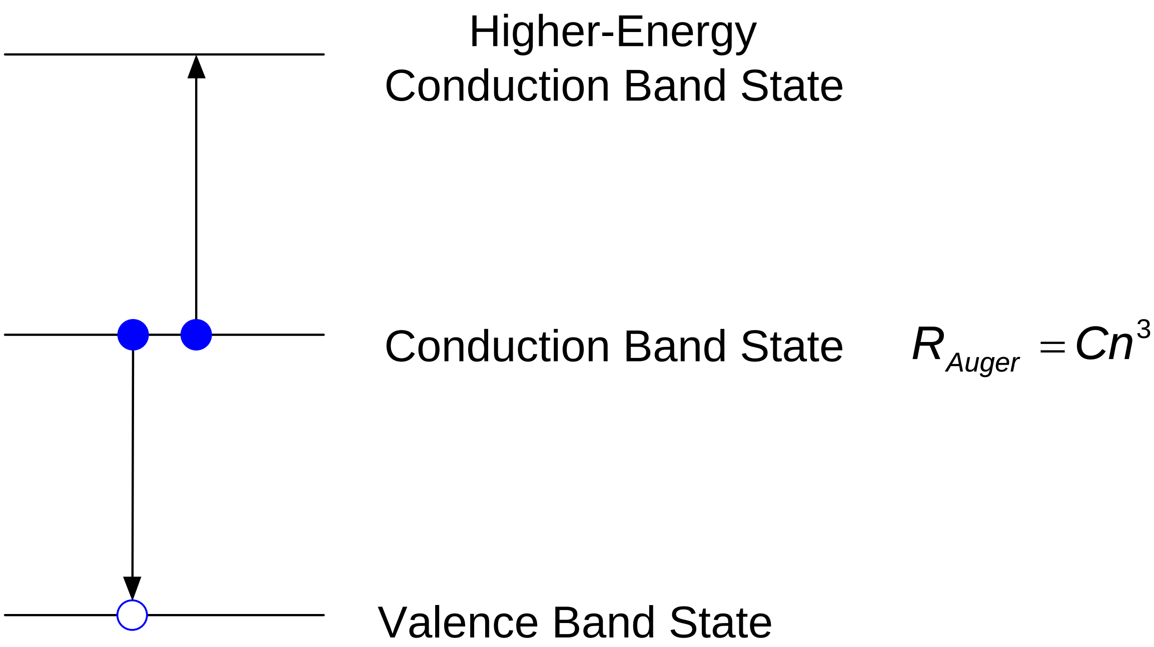

Figure 8.6. *A CCCH Auger recombination process.*

Auger processes are "three body" processes, and the recombination rate is given by

$$R_{Auger} = Cn^3.$$

Auger recombination is particularly important for the operation of light emitting diodes and semiconductor lasers with large electron-hole populations injected in the gain region. Auger recombination also tends to be larger for semiconductors with smaller band gaps such as those used for optical communication sources.

8.1.6 Stimulated Emission

Stimulated emission (**Figure 8.7**) is a recombination process that adds photons to the total photon number ϕ. The stimulated emission rate depends on the number of photons and the optical gain for the circulating optical mode. It also depends indirectly on the density of electron-hole pairs, through the gain coefficient. The electron-hole recombination rate is

$$R_{st} = \Gamma g \frac{c}{n_G} \frac{\phi}{V},$$

where g is the gain coefficient in the gain region, Γ (called the confinement factor) is the fraction of the optical mode that overlaps the gain region, and V is the volume of the gain region.

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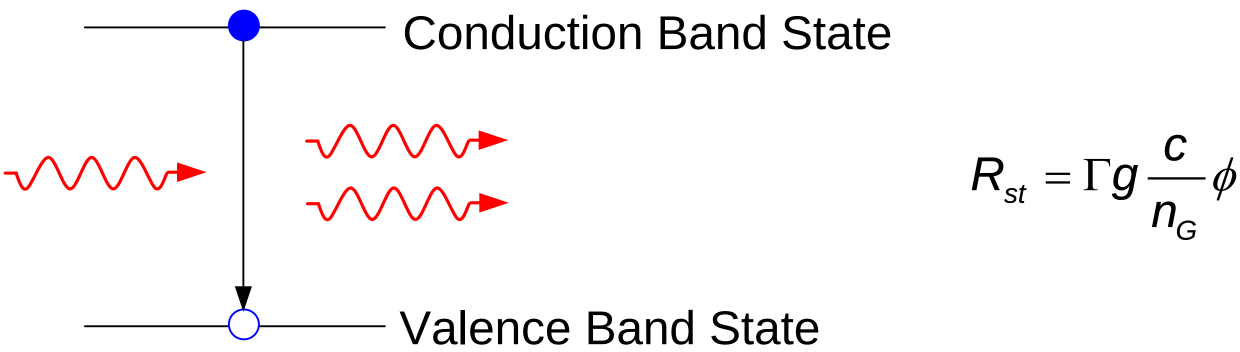

Figure 8.7. *Electron-hole recombination by stimulated emission of a photon.*

8.1.7 Rate Equation for Electron (Hole) Density

Remembering the relation

$$\frac{dn}{dt} = G_{gen} - R_{rec}$$

and using quantities presented in the previous sub-sections, we find

$$\boxed{\frac{dn}{dt} = \frac{\mathrm{I}}{\mathrm{qV}} - A_{nr} n + B n^2 + C n^3 - \Gamma g \frac{c}{n_G} \frac{\phi}{V}}$$

(Equation 8.1.1)

We can also define an effective electron-hole lifetime

$$\frac{1}{\tau_{sp}} = A_{nr} + Bn + Cn^2,$$

so that the rate equation for electron density can be written in a more concise form:

$$\boxed{\frac{dn}{dt} = \frac{\mathrm{I}}{\mathrm{qV}} - \frac{n}{\tau_{sp}} - \Gamma g \frac{c}{n_G} \frac{\phi}{V}}.$$

(Equation 8.1.2)

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8.1.8 Rate Equation for Photon Number

The rate equation for the total photon number is

$$\boxed{\frac{d\phi}{dt} = \underbrace{\Gamma g \frac{c}{n_G}\phi}_{\substack{\text{stimulated}\\\text{emission}}} + \underbrace{\beta B n^2 V}_{\substack{\text{spontaneous}\\\text{emission}}} - \underbrace{\frac{\phi}{\tau_p}}_{\substack{\text{cavity}\\\text{losses}}}}.$$

(Equation 8.2)

The stimulated emission term from **Equations 8.1.1** and **8.1.2** needs only minor modification for inclusion in **Equation 8.2**. First we change the minus sign to a plus sign because stimulated increases the photon number ϕ. Second, we remove the gain region volume V from the term because **Equation 8.2** describes the rate of change of the total photon number, not the photon density.

Like stimulated emission, spontaneous emission contributes to the number of photons in the circulating confined mode. However, since spontaneously emitted photons are emitted in all directions, only a fraction β enter the circulating mode to increase ϕ. The quantity β is called the spontaneous emission factor and its value is on the order of 10^{-5}.

8.1.9 Cavity Losses

In semiconductor lasers, “cavity losses” are of two types. First there is optical loss that is distributed along the length of laser and characterized by a loss coefficient α that has units of inverse length. Background absorption from material imperfections and interaction with dopants, and optical scattering at material interfaces contribute to α. The rate of change of ϕ due to α is

$$\frac{d\phi}{dt} = -\alpha \frac{c}{n_G}\phi,$$

so we have a contribution to the photon cavity lifetime τ_p of **Equation 8.2** (the mean time a photon circulates in the cavity before it is “lost”)

$$\frac{1}{\tau_{p,\alpha}} = \alpha \frac{c}{n_G}.$$

Second, there is discrete optical loss at the laser mirrors of amounts $1\text{-}R_1$ and $1\text{-}R_2$. Since the round trip time for the circulating mode is $c/n_G 2L$, where L is the length of the laser, and ignoring other contributions to loss or gain, we have

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$$\underbrace{\phi\left(\frac{c}{n_G 2L}\right)}_{\substack{\text{After one}\\ \text{round trip}}} = \underbrace{\phi(0)}_{\substack{\text{Initial}\\ \text{Value}}} R_1 R_2 .$$

Furthermore, we can define the contribution of the mirrors to the photon cavity lifetime with

$$\phi\left(\frac{n_G 2L}{c}\right) = \phi(0) e^{-\frac{\frac{n_G 2L}{c}}{\tau_{p,mirrors}}}$$

$$\Rightarrow R_1 R_2 = e^{-\frac{\frac{n_G 2L}{c}}{\tau_{p,mirrors}}} .$$

$$\Rightarrow \frac{1}{\tau_{p,mirrors}} = -\ln(R_1 R_2)\frac{c}{n_G 2L}$$

As a result, we have the following expression for the total cavity lifetime:

$$\frac{1}{\tau_p} = \frac{1}{\tau_{p,\alpha}} + \frac{1}{\tau_{p,mirror}} = \alpha \frac{c}{n_G} - \ln(R_1 R_2)\frac{c}{n_G 2L} .$$

8.2 Analytical Solutions of Simplified Rate Equations

Now that we have our rate equations, we begin our study of laser output by neglecting spontaneous emission in the rate equation for the photon number ϕ. The argument for doing this is that the spontaneous emission is relatively weak and should not create a large number of photons in the laser mode. It will turn out that the approximation is good enough so that the simplified rate equations produce curves of n, g, and ϕ versus I that, at first glance, are very close to the curves produced by a more accurate analysis. On the other hand, this simple analysis predicts that Fabry-Perot-type semiconductor lasers (edge emitting semiconductor lasers without additional frequency selective elements) will operate in just one longitudinal mode – a prediction that is not borne out by experiment. We will see in the last section that inclusion of spontaneous emission is essential for an understanding of the multimode operation of semiconductor lasers.

The simplified rate equations are

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$$\frac{dn}{dt} = \frac{I}{qV} - \frac{n}{\tau_{sp}} - \Gamma g \frac{c}{n_G} \frac{\phi}{V}$$

$$\frac{d\phi}{dt} = \Gamma g \frac{c}{n_G} \phi - \frac{\phi}{\tau_p}$$

When we have steady state conditions, dn/dt = dϕ/dt = 0 and the rate equations become

$$0 = \frac{I}{qV} - \frac{n}{\tau_{sp}} - \Gamma g \frac{c}{n_G} \frac{\phi}{V}$$

(Equation 8.3)

$$0 = \Gamma g \frac{c}{n_G} \phi - \frac{\phi}{\tau_p} = \left(\Gamma g \frac{c}{n_G} - \frac{1}{\tau_p} \right) \phi .$$

(Equation 8.4)

There are two solutions for **Equation 8.4**. One of the solutions,

$$\phi = 0 ,$$

(Equation 8.5)

can be inserted in **Equation 8.3** to give

$$n = I \frac{\tau_{sp}}{qV} .$$

(Equation 8.6)

Equations 8.5 and **8.6** describe the laser pumped below the threshold for lasing. **Equation 8.6** would predict a linear increase of n with increasing I, if τ_{sp} were constant, but the electron-hole recombination rate increases as n increases due to the spontaneous emission and Auger contributions. As a result, **Equation 8.6** predicts a sublinear dependence of n on I.

In general, the gain coefficient is a complicated function of n, but we will use a common approximation:

$$g = a\left(n - n_{tr}\right),$$

(Equation 8.7)

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where a is called the gain constant and n_{tr} is called the carrier density at transparency. In this approximation, g also increases sublinearly with increasing current I.

Now we consider the other solution to **Equation 8.4**:

$$\left(\Gamma g \frac{c}{n_G} - \frac{1}{\tau_p}\right) = 0 \Rightarrow g = \frac{n_G}{\Gamma c \tau_p}.$$

(Equation 8.8)

Equation 8.8 describes the laser above threshold (i.e. lasing). The important thing to notice anout **Equation 8.8** is that the gain coefficient g is constant or “clamped” above threshold. **Equation 8.8** shows that the electron-hole density is also clamped. Using **Equation 8.8** in **Equation 8.3** gives

$$\phi = I \frac{\tau_p}{q} - \frac{n \tau_p V}{\tau_{sp}}.$$

(Equation 8.9)

The electron-hole recombination time τ_{sp} depends only on n, so τ_{sp} is also constant above the lasing threshold, so **Equation 8.9** shows that ϕ increases linearly with the injection current. The laser output is proportional to ϕ, so it also increases linearly with current I. The results of this section are summarized by the graphs in **Figure 8.8**.

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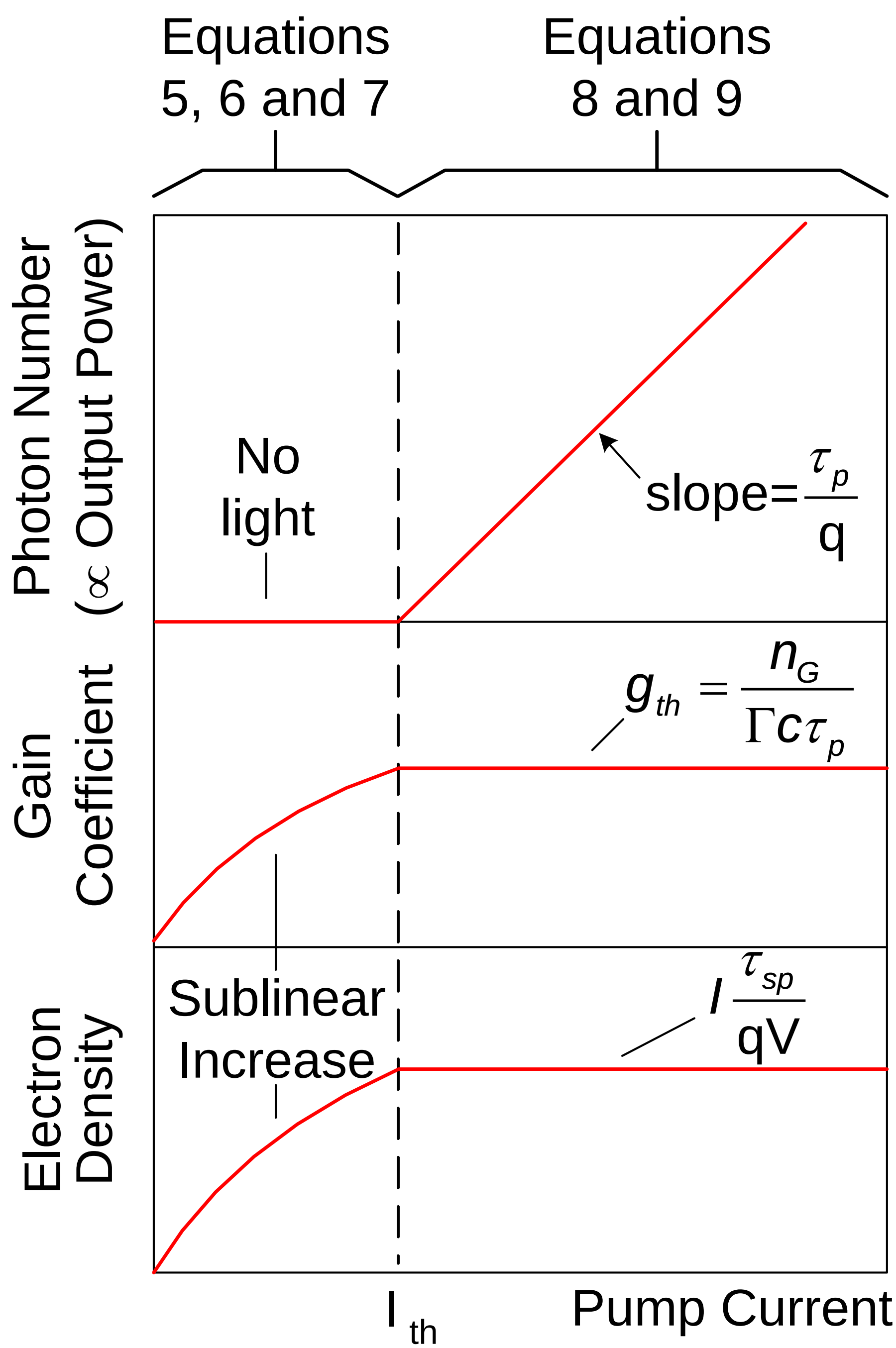

Figure 8.8. *Analytical solutions to simplified rate equations, neglecting the contribution of spontaneous emission to the equation for photon number.*

The implication of the results summarized in **Figure 8.8** is that the laser can operate in only a single longitudinal mode. To see that this is so, consider **Figure 8.9**. The laser medium is homogeneously broadened, so all longitudinal modes see the same gain curve. A particular mode lases and contributes optical output when gain balances loss at the corresponding wavelength. For all other modes, the gain is below this threshold value, and as we can see from **Figure 8.8**, this implies zero photons in the cavity and no laser output.

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One additional note: There is a less likely situation in which the previous argument allows for two modes to lase. It is possible for two modes located near the gain peak and on either side of the gain peak to have the same gain. In this case it is also possible for both modes to have gain that balances loss.

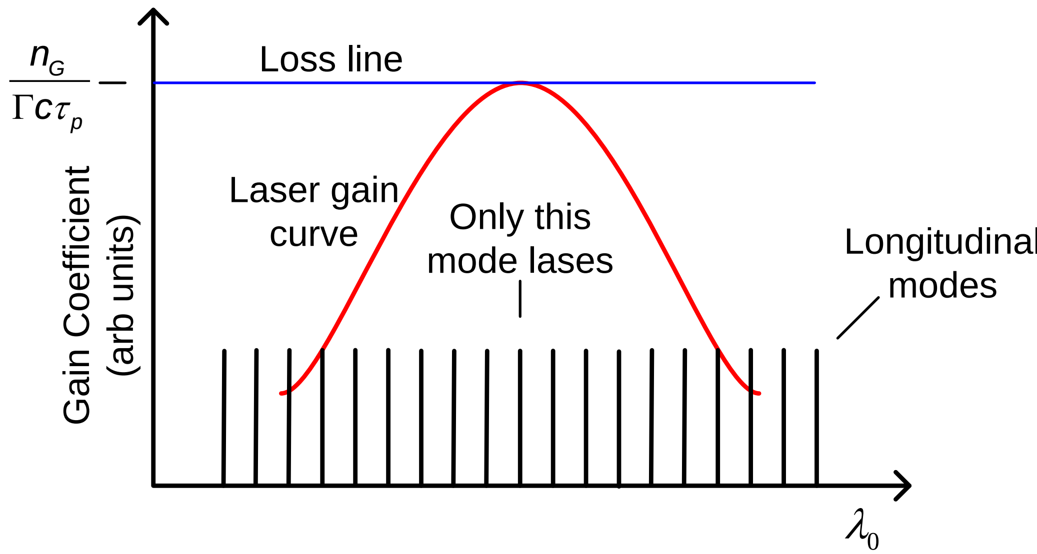

Figure 8.9. *The laser operates in a single longitudinal mode at a wavelength for which gain equal loss.*

8.3 Numerical Solutions for the Rate Equations: Multimode Operation

In this section we will numerical solve the rate equations, including spontaneous emission in both equations, for a more precise examination of the dependence of n, g, and ϕ on I. The changes to the results presented in **Figure 8.8** will appear to be subtle. However, we will find that there is no precise threshold for the laser – a finding that will lead us to an understanding of the multimode operation of semiconductor lasers.

We begin with the rate equations (**Equations 8.1** and **8.2**), this time leaving the spontaneous emission term in the equation for dϕ/dt. We will use the linear approximation for the gain coefficient (**Equation 8.7**) and the values in Table 1.

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Table 1. Parameters for a typical semiconductor laser.

Parameter	Symbol	Value
Laser length	L	250 μm
Active region width	w	2 μm
Active region height	h	0.2 μm
Confinement factor	Γ	0.3
Group refractive index	n_G	4
Gain constant	a	2.5×10^{-16} cm^2
Carrier density at transparency	n_{tr}	1×10^{18} cm^{-3}
Nonradiative recombination constant	A_{nr}	1×10^{-18} s^{-1}
Spontaneous emission constant	B	1×10^{-10} cm^3/s
Auger constant	C	1×10^{-29} cm^6/s
Photon cavity lifetime	τ_p	1.6 ps
Spontaneous emission factor	β	10^{-5}

The calculations were carried out using a MathCad software routine that is listed below for the reader who wishes to reproduce the calculation and perhaps see the effects of changing values in Table 1. Numerical solutions for n, g, and ϕ are pictured in **Figure 8.10**.

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MathCad Software Routine for Solving the Rate Equations

$$q := 1.6\cdot 10^{-19} \qquad V := 250\cdot 10^{-4}\cdot 2\cdot 10^{-4}\cdot .2\cdot 10^{-4}$$

$$Anr := 1\cdot 10^{8} \qquad B := 1\cdot 10^{-10} \qquad C := 3\cdot 10^{-29}$$

$$c := 3\cdot 10^{10} \qquad ng := 4 \qquad \Gamma := .3 \qquad a := 2.5\cdot 10^{-16} \qquad nt := 1\cdot 10^{18}$$

$$\beta := 10^{-5} \qquad \tau p := 1.6\cdot 10^{-12}$$

$$TOL := 10^{-8}$$

$$i := 0, 1 .. 211111 \qquad n_i := i\cdot 10^{13}$$

$$I := .01$$

$$I_i := \mathrm{root}\left[I - q\cdot V\cdot\left[Anr\cdot n_i + B\cdot\left(n_i\right)^2 + C\cdot\left(n_i\right)^3 + \frac{c}{ng}\cdot\left[a\cdot\left(n_i - nt\right)\cdot\Gamma\cdot\left[\frac{\beta\cdot B\cdot\left(n_i\right)^2}{\frac{1}{\tau p} - \frac{c}{ng}\cdot\Gamma\cdot a\cdot\left(n_i - nt\right)}\right]\right]\right], I\right]$$

$$\phi_i := \frac{(\beta\cdot B)\cdot\left(n_i\right)^2\cdot V}{\frac{1}{\tau p} - \frac{c}{ng}\cdot\Gamma\cdot a\cdot\left(n_i - nt\right)} \qquad g_i := a\cdot\left(n_i - nt\right)$$

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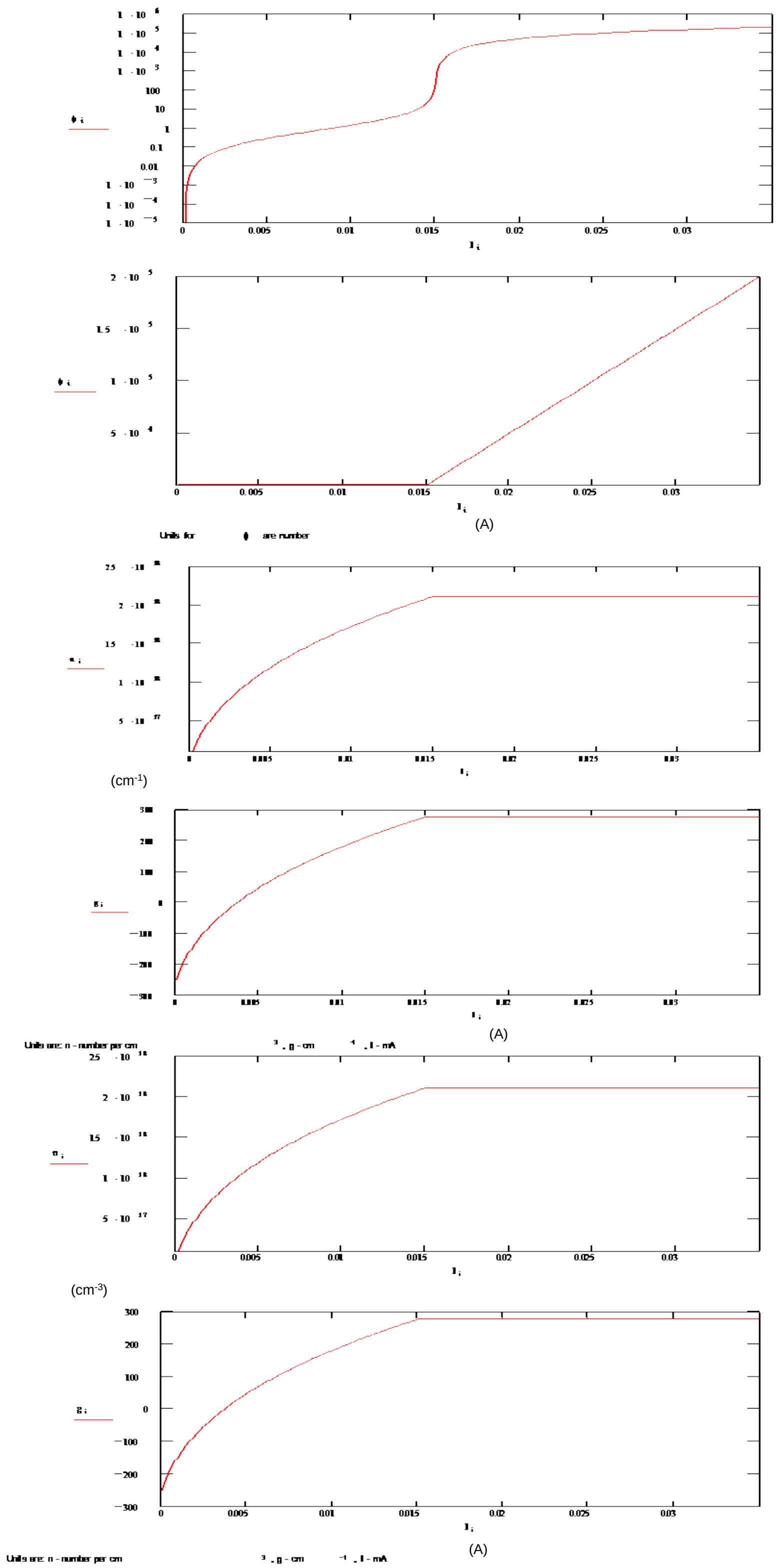

Figure 8.10. *The electron density, gain coefficient, and total photon number simulated with the rate equations, Equation 8.7, and the values in Table 1.*

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Figure 8.10 looks very much like **Figure 8.8**; however, a closer look at the carrier density and gain coefficient (**Figure 8.11**) shows that these quantities are not clamped above a threshold value.

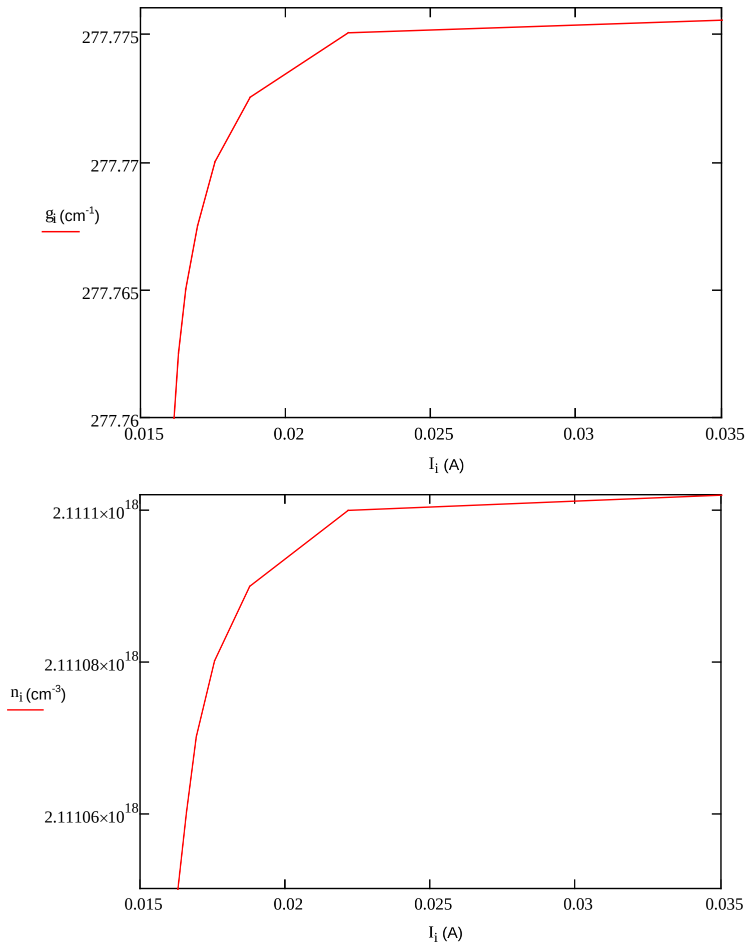

Figure 8.11. *A close-up view of carrier density and gain coefficient at higher pump currents.*

Another significant difference in the results can be seen in **Figure 8.12**, where the total photon number is plotted on a log scale. Notice that there is a nonzero number of photons in the laser mode for all pumping currents, which we can attribute to amplified spontaneous emission.

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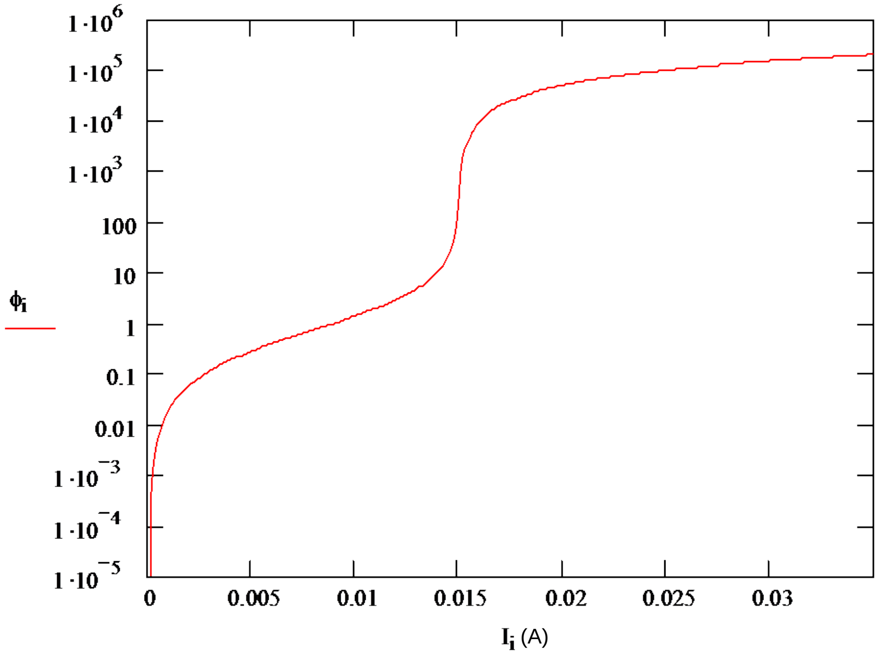

Figure 8.12. *Total photon number as a function of injection current.*

These results are what are needed to understand the origin of multimode operation in semiconductor lasers. There is no precise threshold for lasing in the conventional sense. All longitudinal modes experience gain that is less than the cavity losses, but amplified spontaneous emission continuously adds photons to the modes, producing laser output. Modes at wavelengths near the peak of the gain curve have the highest output.

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