

- **FDTD for 1D wave equation**

- **Equation:** $\frac{\partial^2 H}{\partial t^2} = c^2 \frac{\partial^2 H}{\partial x^2}$

- **Notations:**

- $t = n\Delta t, x = i\Delta x$

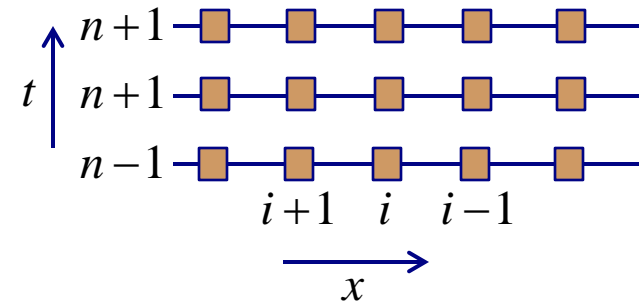
- $H(n\Delta t, i\Delta x) = H_i^n$

- $E(n\Delta t, i\Delta x) = E_i^n$

- **discretization**

$$\frac{H_i^{n+1} - 2H_i^n + H_i^{n-1}}{(\Delta t)^2} = c^2 \frac{H_{i+1}^n - 2H_i^n + H_{i-1}^n}{(\Delta x)^2}$$

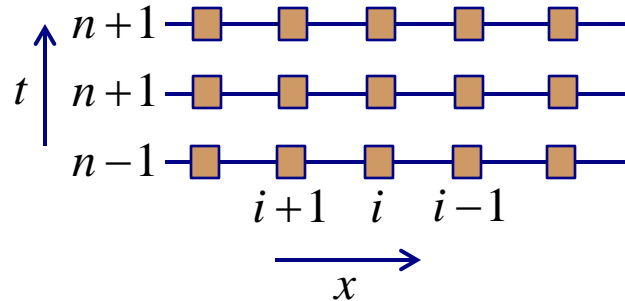
$$\Rightarrow H_i^{n+1} = \underbrace{\left(\frac{c\Delta t}{\Delta x}\right)^2}_{\gamma^2} (H_{i+1}^n + H_{i-1}^n) + 2(1 - \gamma^2)H_i^n - H_i^{n-1}$$



- **FDTD for 1D wave equation (cont'd)**

- **Properties**

- Fully explicit second-order accurate scheme
- Two previous time steps are required to be stored
- The analysis is recursive in time
- Each field is found at every time step at every node



- **FDTD for 1D Maxwell's equations**

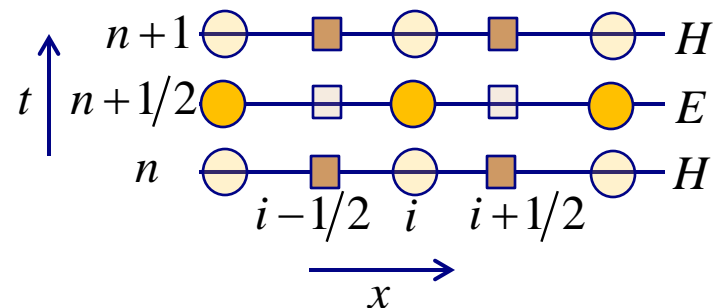
- **Equation:**
$$\frac{\partial H}{\partial t} = \frac{1}{\mu} \frac{\partial E}{\partial x}; \quad \frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \frac{\partial H}{\partial x}$$

- **Notations:**

- Redefine: $2\Delta t \rightarrow \Delta t$
- Central difference: $\frac{\partial H}{\partial t} \approx \frac{H(t + \Delta t/2) - H(t - \Delta t/2)}{\Delta t}$
- Involved time steps: $t = n\Delta t, t = n\Delta t + \Delta t/2$
- Involved spatial steps: $x = i\Delta x, x = i\Delta x + \Delta x/2$
- Involved fields:

$$H(n\Delta t, i\Delta x + \Delta x/2) = H_{i+1/2}^n$$

$$E(n\Delta t + \Delta t/2, i\Delta x) = E_i^{n+1/2}$$

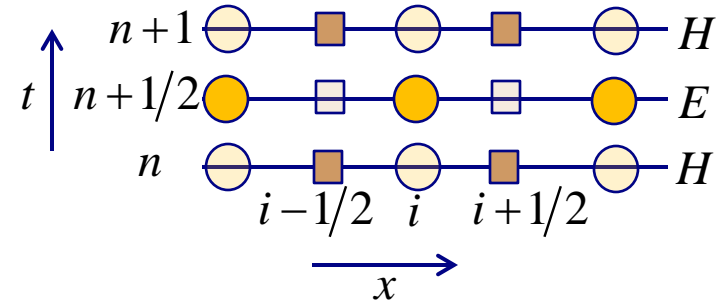


- **FDTD for 1D Maxwell's equations (cont'd)**

- **Update equations:**

$$H_{i+1/2}^{n+1} = H_{i+1/2}^n + \frac{\Delta t}{\Delta x} \frac{1}{\mu} (E_{i+1}^{n+1/2} - E_i^{n+1/2})$$

$$E_i^{n+1/2} = E_i^{n-1/2} + \frac{\Delta t}{\Delta x} \frac{1}{\varepsilon} (H_{i+1/2}^n - H_{i-1/2}^n)$$



- **Yee algorithm:**

- The fields are given in different space and time locations
- Interleaved E & H grids
- Leapfrog in time
- Explicit stepping → no matrix inversions, no iterations
- The solution is more robust and straightforward
- E & H boundary conditions are enforced automatically

Numerical dispersion of FDTD

- Consider a time harmonic wave $H = e^{j(\omega t - kx)}$
- Continuous dispersion relation is linear: $\omega = ck$
- Numerical wave: $H_i^n = e^{j(\omega n \Delta t - ki \Delta x)}$
- Use the FDTD update equation $H_i^{n+1} = \gamma^2 (H_{i+1}^n + H_{i-1}^n) + 2(1 - \gamma^2) - H_i^{n-1}$
- $\rightarrow e^{j\omega \Delta t} = \gamma^2 (e^{-jk \Delta x} + e^{jk \Delta x}) + 2(1 - \gamma^2) - e^{-j\omega \Delta t}$
- $\rightarrow \boxed{\sin^2\left(\frac{\omega \Delta t}{2}\right) = \gamma^2 \sin^2\left(\frac{k \Delta x}{2}\right)}$
- **Properties:**
 - In general the phase velocity depends on the frequency
 \rightarrow numerical dispersion!
 - $\Delta t, \Delta x \rightarrow 0 \rightarrow \omega^2 = (ck)^2$ - weak dispersion
 - $\Delta x = c\Delta t, \gamma = 1 \rightarrow$ “magic” time step with no dispersion (for the 1D case)

Stability of FDTD

- Consider $H = e^{j(\omega t - kx)}$ and $H_i^n = e^{j(\omega n \Delta t - ki \Delta x)}$
- Allow ω to be complex, i.e. $\omega = \omega' + j\omega''$
- The propagation is stable if $\omega'' \geq 0$ (wave $\sim e^{-|\omega'' t|}$)
- For $\omega'' < 0$ (wave $\sim e^{|\omega'' t|}$) the propagation is unstable
- Define $\Omega = \omega \Delta t / 2$, $K = k \Delta x / 2$, $g = e^{-j\Omega}$ and use $\sin \Omega - \gamma \sin K = 0$
- $\rightarrow g^2 + 2j\gamma g \sin K - 1 = 0 \Rightarrow g = -jK \sin K \pm \sqrt{1 - \gamma^2 \sin^2 K}$
- **Stable solution:** $|g| \leq 1 \Rightarrow 1 - \gamma^2 \sin^2 K \geq 0 \Rightarrow \gamma \leq 1$
- **Courant stability criterion for 1D case:**

$$\Delta t \leq \frac{\Delta x}{c}$$

- **Stability of FDTD (cont'd)**

- **Physical meaning: the wave should not pass through more than one cell in a single time step**
- **Under the magic time step the scheme is exact**

- **How to choose Δx and Δt**

- **If no fine geometrical features are present:**
 $\Delta x < \lambda_{\min}/10$ to satisfy Nyquist sampling criterion
- **If fine geometrical features are present:**
 Δx is determined by the features
- **Choose $\Delta t < \Delta x/c$ (note that for fine geometrical features, Δx is small and the time step is small as well!)**

- **Source modeling**

- **Initial conditions: Define E_i^0, H_i^1 for all i .**

- **Hard source**

- **Specify fields at certain locations independent of surrounding cells, i.e. replace the Yee equations, e.g.**

$$E_{z,i_s}^n = F(t_n)$$

- **Easy to implement but lead to reflections**

- **Soft source**

- **Sources that allow field updates, e.g. current sources**

- **Equations are updated and no reflections but harder to implement**

- **Plane wave injection: Scattered field / total field approach**

- **Time dependence of sources**

- **Time harmonic sources**

- **Simple sin functions leads to discontinuities and a relatively broadband spectrum**
- **Use sin with a smooth transition function ($g(t)$)**

$$F(t) = F_0 \sin(\omega_0 n \Delta t) g(t)$$

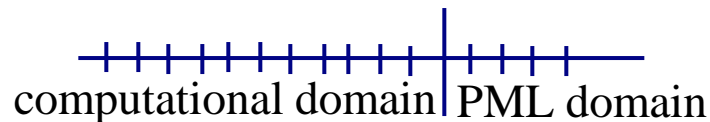
- **Pulsed sources**

- **Gaussian: Contains DC → a problem**
- **Time derivative of a Gaussian → no DC**
- **Modulated Gaussian**

- **Domain truncation: Perfectly matched layer (PML)**

- Add an additional domain where a truncation is needed
- Fill the domain with a matching lossy material with σ & σ_e satisfying

$$\frac{\sigma_m}{\mu_1} = \frac{\sigma}{\varepsilon_1} \quad \& \quad \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$



- **→ No reflections are obtained because**

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2} \frac{1 + \sigma_m / (j\omega\mu_2)}{1 + \sigma / (j\omega\varepsilon_2)}}$$

- This idea can be extended to 2D and 3D but the lossy materials should be anisotropic to match all directions
- Other absorbing boundary conditions can be used as well

3-D FDTD Algorithm

Maxwell's equations:

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial \mathcal{H}_x}{\partial t}$$

$$\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} = \varepsilon \frac{\partial \mathcal{E}_x}{\partial t} + \sigma \mathcal{E}_x + \mathcal{J}_x$$

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\mu \frac{\partial \mathcal{H}_y}{\partial t}$$

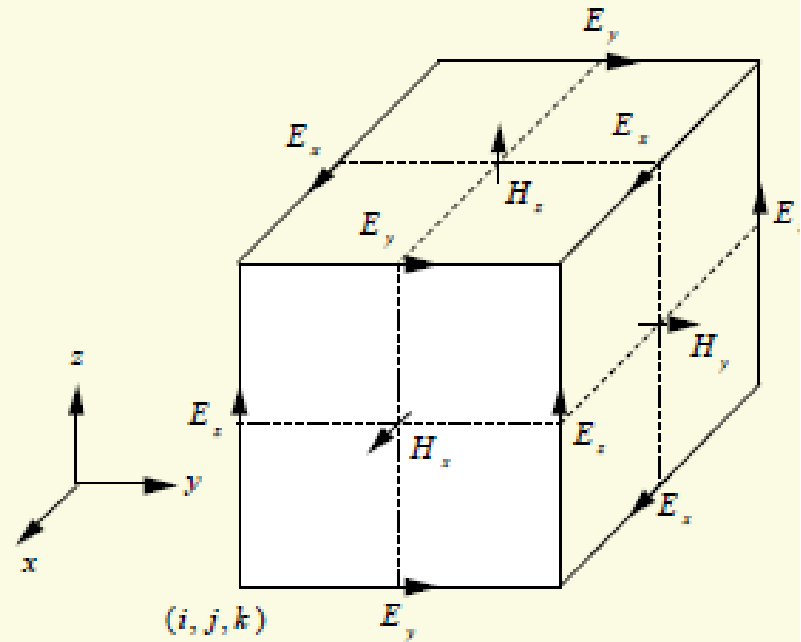
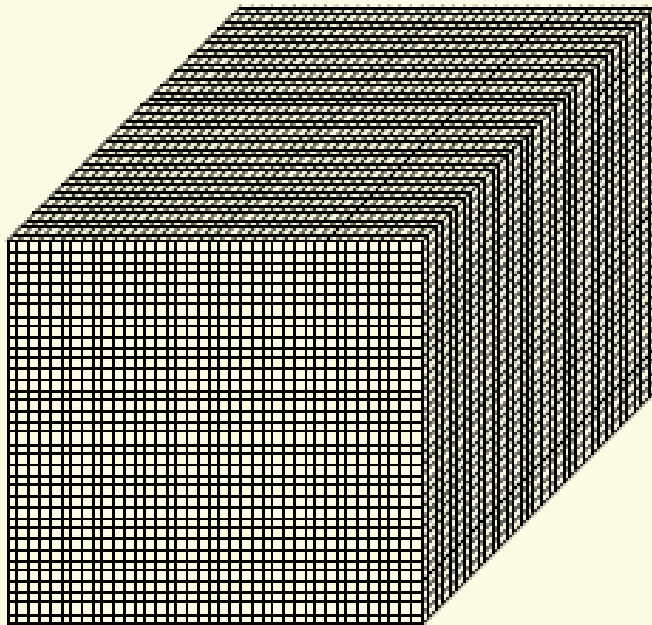
$$\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} = \varepsilon \frac{\partial \mathcal{E}_y}{\partial t} + \sigma \mathcal{E}_y + \mathcal{J}_y$$

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\mu \frac{\partial \mathcal{H}_z}{\partial t}$$

$$\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = \varepsilon \frac{\partial \mathcal{E}_z}{\partial t} + \sigma \mathcal{E}_z + \mathcal{J}_z$$

3-D FDTD Algorithm

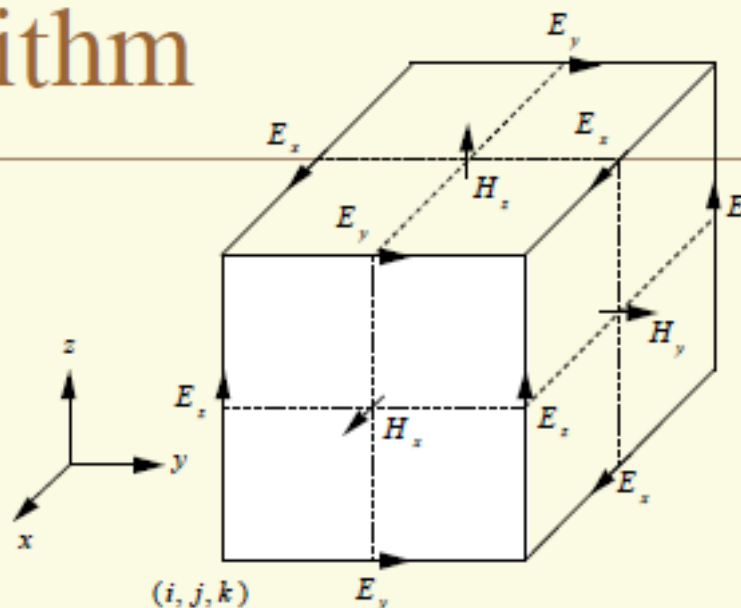
FDTD discretization and Yee cell:



3-D FDTD Algorithm

FDTD discretization:

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial \mathcal{H}_x}{\partial t}$$



$$\frac{\mathcal{E}_z^n(i, j+1, k+\frac{1}{2}) - \mathcal{E}_z^n(i, j, k+\frac{1}{2})}{\Delta y}$$

$$- \frac{\mathcal{E}_y^n(i, j+\frac{1}{2}, k+1) - \mathcal{E}_y^n(i, j+\frac{1}{2}, k)}{\Delta z}$$

$$= -\mu \frac{\mathcal{H}_x^{n+\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) - \mathcal{H}_x^{n-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2})}{\Delta t}$$

3-D FDTD Algorithm

Stability condition:

$$\Delta t \leq \frac{1}{c \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}$$

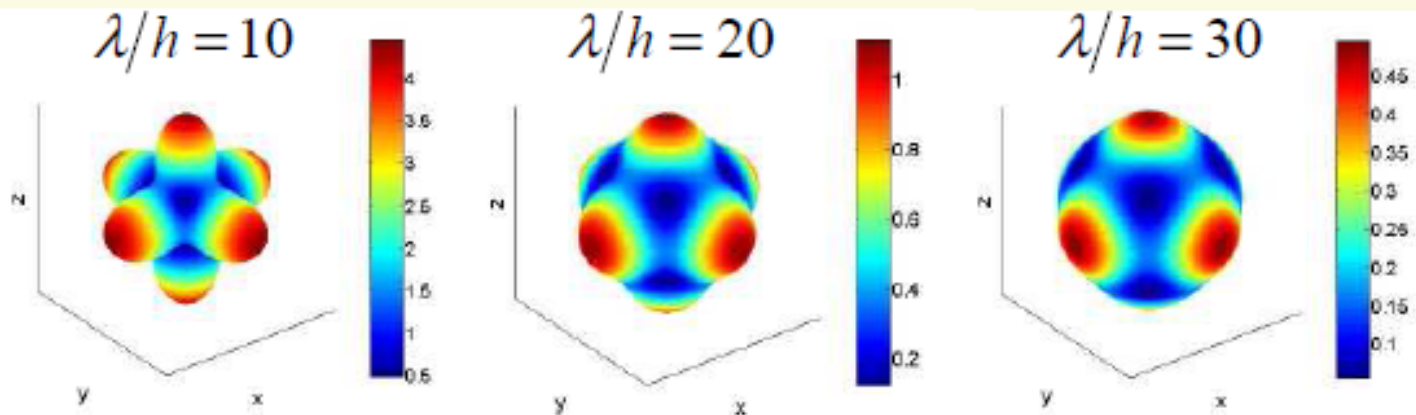
Numerical dispersion:

$$\frac{\tilde{k} - k}{k} \approx \frac{1}{24} \left\{ [(k\Delta x)^2 \cos^4 \phi^i + (k\Delta y)^2 \sin^4 \phi^i] \sin^4 \theta^i + (k\Delta z)^2 \cos^4 \theta^i - (\omega\Delta t)^2 \right\}$$

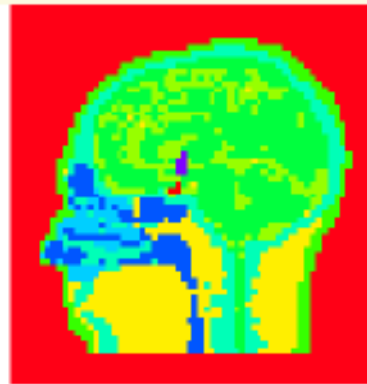
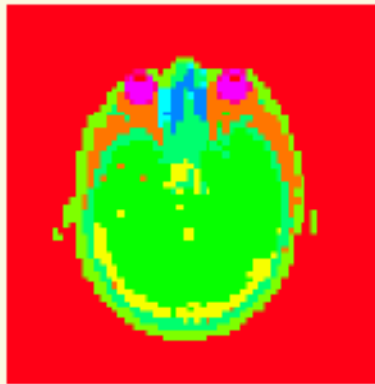
3-D FDTD Algorithm

Numerical dispersion error for $\Delta t = 0.5h/c$:

$$\frac{\tilde{k} - k}{k} \approx \frac{\pi^2}{6} \left(\frac{h}{\lambda} \right)^2 \left\{ [\cos^4 \phi^i + \sin^4 \phi^i] \sin^4 \theta^i + \cos^4 \theta^i - \frac{1}{4} \right\}$$



3-D FDTD Example



4 mm resolution

14 kinds of tissues

Outline

- Introduction
- Maxwell's equations
- Finite Difference Time Domain method
- Finite element method
- Integral equation methods

Integral equation procedure

- Formulate an integral equation
- Expand the unknowns in terms of basis functions
- Define a discrete matrix representation of the continuous integral equation
- Solve the matrix equation
- Post-process to derive the required parameters

Advantages

- Only the structure of interest is discretized
- Outgoing conditions are satisfied automatically
- No numerical dispersion
- Can be formulated in time and frequency
- Can be defined for surfaces and volumes
- Can be formulated for complex backgrounds
- Can be higher order accurate

Challenges

- Computational cost of “simple” methods is $\sim O(N^2)$
- \rightarrow Need for fast methods ($\sim O(N)$ or $O(N \log N)$) for N degrees of freedom
- Different formulations are required for different problem types, e.g. volumes vs. surfaces

Equation of electrostatics

- **Electrostatics**

- $\partial/\partial t = \omega = 0 \Rightarrow$ **two uncoupled sets of (static) equations are obtained for \mathbf{E} & \mathbf{D} and \mathbf{H} & \mathbf{B}**
- **Equation of electrostatics**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

or

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

- **Boundary conditions**

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

- **Constitutive relation** $\mathbf{D} = \varepsilon \mathbf{E}$

Electric scalar potential

- Potential due to various charge distributions

- Point charge

$$V(\mathbf{R}) = \frac{q}{4\pi\epsilon|\mathbf{R} - \mathbf{R}_1|}$$

- Many discrete charges

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon} \sum_{i=1}^N \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|}$$

- Various continuous charge distributions in free space

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho_v}{R'} dV' \quad \text{(volume distribution),} \quad (4.48a) \quad \overline{\overline{\overline{R'}}} = |\mathbf{R} - \mathbf{R}_i|$$

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon} \int_{S'} \frac{\rho_s}{R'} ds' \quad \text{(surface distribution),} \quad (4.48b)$$

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon} \int_{L'} \frac{\rho_l}{R'} dl' \quad \text{(line distribution).} \quad (4.48c)$$

Capacitance

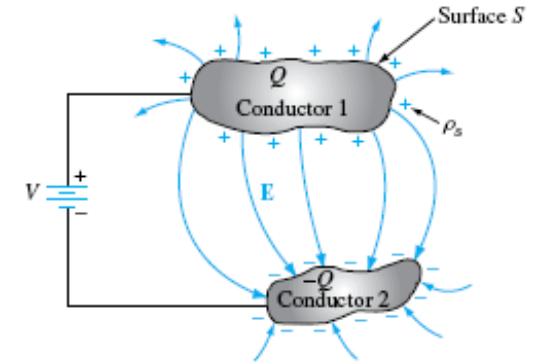
- Consider a capacitor consisting of two conductors with a voltage
- Opposite charges appear on the conductors

- Capacitance $C = \frac{Q}{V}$ (C/V or F),

- Alternative expression

$$Q = \int_S \rho_s ds = \int_S \epsilon \hat{n} \cdot \mathbf{E} ds = \int_S \epsilon \mathbf{E} \cdot d\mathbf{s},$$

$$V = V_{12} = - \int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l},$$



$$\Rightarrow C = \frac{\int_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{- \int_l \mathbf{E} \cdot d\mathbf{l}}$$

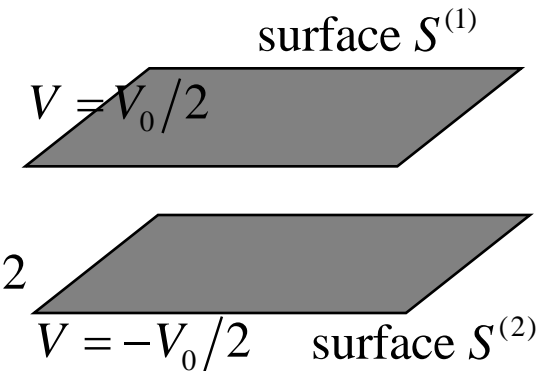
- Relation to resistance $RC = \frac{\epsilon}{\sigma}$

Numerical capacitance extraction (1)

- Integral equation method

- Given

- Two conductors
- Voltage on the conductors is $V_0/2$ and $-V_0/2$
- Find the capacitance C



- Formulation

- Potential on the surfaces is given by $V_0/2$ and $-V_0/2$
- Potential everywhere is calculated as $V(\mathbf{r}) = \int_S \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \underbrace{\rho_s(\mathbf{r}')}_{\text{surface charge distribution}} dS'$
where ρ_s is an unknown surface charge
- Equate the potentials on the surfaces

$$\Rightarrow \iint_{S^{(1)} + S^{(2)}} \underbrace{\frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}}_{\text{Green's function}} \underbrace{\rho_s(\mathbf{r}')}_{\substack{\text{surface charge} \\ \text{distribution} \\ \text{(unknown)}}} dS' = \begin{cases} V_0/2; \mathbf{r} \in S_1 \\ -V_0/2; \mathbf{r} \in S_2 \end{cases} \leftarrow \text{integral equation}$$

Numerical capacitance extraction (2)

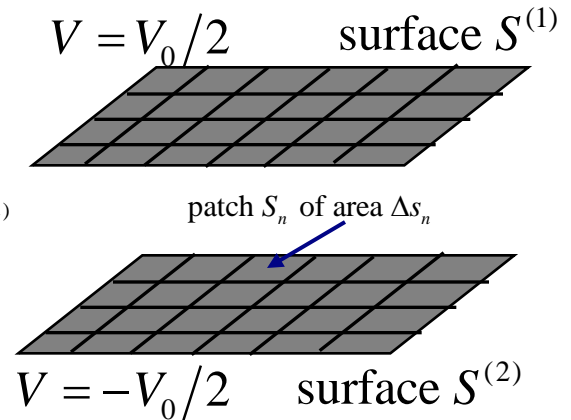
- Matrix equation

- Discretize the problem

- Divide the surfaces into $N = N_{S^{(1)}} + N_{S^{(2)}}$ small patches, $N_{S^{(1)}}, N_{S^{(2)}}$ are numbers of patches on surfaces $S^{(1)}, S^{(2)}$

- Replace the integration by summation
 - Enforce the IE at the location of charges

- $$\Rightarrow \sum_{n=1}^N \iint_{S_n} \frac{\rho_s(\mathbf{r}') ds'}{4\pi\epsilon_0 |\mathbf{r}_m - \mathbf{r}'|} = \begin{cases} V_0/2; & \mathbf{r}_m \in S_1 \\ -V_0/2; & \mathbf{r}_m \in S_2 \end{cases}$$



- Obtain a matrix equation

$$\sum_{n=1}^N \iint_{S_n} \frac{\rho_s(\mathbf{r}') ds'}{4\pi\epsilon_0 |\mathbf{r}_m - \mathbf{r}'|} = \begin{cases} V_0/2_0; & \mathbf{r}_m \in S_1 \\ -V_0/2; & \mathbf{r}_m \in S_2 \end{cases} \Rightarrow \sum_{n=1}^N Z_{mn} Q_n = V_m \Rightarrow \boxed{\underbrace{\underline{ZQ}}_{\text{matrix equation}} = \underline{V}}$$

Numerical capacitance extraction (3)

- Matrix equation and solution

— Matrix equation $\underline{\underline{Z}}\underline{\underline{Q}} = \underline{\underline{V}}$

$$\underline{\underline{Z}} : N \times N \text{ matrix}; Z_{mn} = \frac{1}{\Delta S_n} \iint_{S_n} \frac{ds'}{4\pi\epsilon_0 |\mathbf{r}_m - \mathbf{r}'|} \Rightarrow \begin{cases} Z_{mn} \approx \frac{1}{4\pi\epsilon_0 |\mathbf{r}_m - \mathbf{r}_n|}; \mathbf{r}_m \neq \mathbf{r}_n \\ Z_{mn} \approx \frac{1}{2\epsilon_0 \sqrt{\pi \Delta S_n}}; \mathbf{r}_m = \mathbf{r}_n \end{cases}$$

$$\underline{\underline{Q}} : N \text{ vector}; Q_n = \rho_s(\mathbf{r}_n) \Delta S_n$$

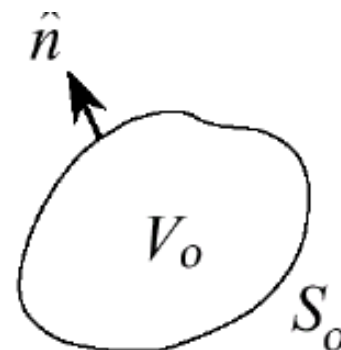
$$\underline{\underline{V}} : N \text{ vector}; V_m = \begin{cases} V_0/2; \mathbf{r}_m \in S_1 \\ -V_0/2; \mathbf{r}_m \in S_2 \end{cases}$$

— Solution $\underline{\underline{Q}} = \underline{\underline{Z}}^{-1} \underline{\underline{V}}$

- Extracted capacitance

$$C = \frac{\overbrace{Q_{S^{(1)}}}^{\text{total charge on surface 1}}}{V_0} \Rightarrow C = \frac{\sum_{n=1}^{N_{S^{(1)}}} Q_n}{V_0}$$

3D Integral Equations



Electric field integral equation (EFIE):

$$\frac{1}{2} \mathbf{M}_s(\mathbf{r}) - \hat{n} \times \mathcal{L}(\bar{\mathbf{J}}_s) + \hat{n} \times \tilde{\mathcal{K}}(\mathbf{M}_s) = -\hat{n} \times \mathbf{E}^{\text{inc}}(\mathbf{r}) \quad \mathbf{r} \in S_o$$

Magnetic field integral equation (MFIE):

$$\frac{1}{2} \bar{\mathbf{J}}_s + \hat{n} \times \mathcal{L}(\mathbf{M}_s) + \hat{n} \times \tilde{\mathcal{K}}(\bar{\mathbf{J}}_s) = \hat{n} \times \bar{\mathbf{H}}^{\text{inc}}(\mathbf{r}) \quad \mathbf{r} \in S_o$$

For a conducting object: $\hat{n} \times \mathbf{E}(\mathbf{r}) = 0 \quad \mathbf{r} \in S_o \Rightarrow \mathbf{M}_s = 0$

EFIE: $\hat{n} \times \mathcal{L}(\bar{\mathbf{J}}_s) = \hat{n} \times \mathbf{E}^{\text{inc}}(\mathbf{r}) \quad \mathbf{r} \in S_o$

MFIE: $\frac{1}{2} \bar{\mathbf{J}}_s + \hat{n} \times \tilde{\mathcal{K}}(\bar{\mathbf{J}}_s) = \hat{n} \times \bar{\mathbf{H}}^{\text{inc}}(\mathbf{r}) \quad \mathbf{r} \in S_o$

3D Integral Equations

Introduce operators:

$$\mathcal{L}(\mathbf{X}) = jk_0 \oiint_{S_o} \left[\mathbf{X}(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') + \frac{1}{k_0^2} \nabla' \cdot \mathbf{X}(\mathbf{r}') \nabla G_0(\mathbf{r}, \mathbf{r}') \right] dS'$$

$$\mathcal{K}(\mathbf{X}) = \oiint_{S_o} \mathbf{X}(\mathbf{r}') \times \nabla G_0(\mathbf{r}, \mathbf{r}') dS'$$

Introduce equivalent currents:

$$\bar{\mathbf{J}}_s(\mathbf{r}') = \hat{n}' \times \bar{\mathbf{H}}(\mathbf{r}') = Z_0 \hat{n}' \times \mathbf{H}(\mathbf{r}') \quad \mathbf{M}_s(\mathbf{r}') = \mathbf{E}(\mathbf{r}') \times \hat{n}'$$

$$\mathbf{E}^{\text{inc}}(\mathbf{r}) - \mathcal{L}(\bar{\mathbf{J}}_s) + \mathcal{K}(\mathbf{M}_s) = \begin{cases} \mathbf{E}(\mathbf{r}) & \mathbf{r} \in V_\infty \\ 0 & \mathbf{r} \in V_o \end{cases}$$

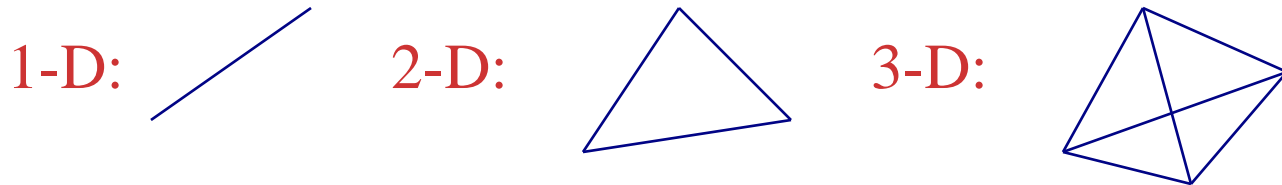
$$\bar{\mathbf{H}}^{\text{inc}}(\mathbf{r}) - \mathcal{L}(\mathbf{M}_s) - \mathcal{K}(\bar{\mathbf{J}}_s) = \begin{cases} \bar{\mathbf{H}}(\mathbf{r}) & \mathbf{r} \in V_\infty \\ 0 & \mathbf{r} \in V_o \end{cases}$$

Outline

- Introduction
- Maxwell's equations
- Finite Difference Time Domain method
- **Finite element method**
- Integral equation methods

Basic FEM Steps

1. Discretization/subdivision of solution domain



2. Selection of interpolation schemes

Linear or higher-order polynomials

3. Formulation of the system of equations

Using either the Ritz or Galerkin method:
Formulate elemental equations and assemble

4. Solution of the system of equations

Using either a direct or iterative method

Advantages

- Works for general structures
- Can be formulated in time and frequency
- Can be higher order accurate
- Low cost ($\sim O(N)$)

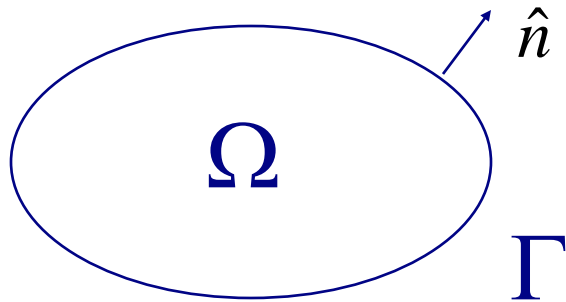
Boundary-Value Problem

Partial differential equation (PDE):

$$-\nabla \cdot (\alpha \nabla \phi) + \beta \phi = f \quad (x, y) \in \Omega$$

Boundary condition (BC):

$$\phi = p \quad \text{on } \Gamma$$



Equivalent Variational Problem

Equivalent variational problem:

$$\delta F(\phi) = 0$$

where

$$F(\phi) = \frac{1}{2} \iint_{\Omega} [\alpha \nabla \phi \cdot \nabla \phi + \beta \phi^2] d\Omega - \iint_{\Omega} f \phi d\Omega$$

Equivalent Variational Problem

Proof:

Taking the first variation:

$$\delta F(\phi) = \iint_{\Omega} [\alpha \nabla \phi \cdot \nabla \delta \phi + \beta \phi \delta \phi] d\Omega - \iint_{\Omega} f \delta \phi d\Omega$$

Invoking the identity

$$\alpha \nabla \phi \cdot \nabla \delta \phi = \nabla \cdot (\alpha \delta \phi \nabla \phi) - \delta \phi \nabla \cdot (\alpha \nabla \phi)$$

$$\begin{aligned} \delta F(\phi) &= \iint_{\Omega} \nabla \cdot [\alpha \delta \phi \nabla \phi] d\Omega + \iint_{\Omega} [-\nabla \cdot (\alpha \nabla \phi) + \beta \phi] \delta \phi d\Omega \\ &\quad - \iint_{\Omega} f \delta \phi d\Omega \end{aligned}$$

Equivalent Variational Problem

Applying the divergence theorem

$$\iint_{\Omega} \nabla \cdot \mathbf{a} \, d\Omega = \oint_{\Gamma} \mathbf{a} \cdot \hat{\mathbf{n}} \, d\Gamma$$

$$\begin{aligned} \delta F(\phi) = & \iint_{\Omega} [-\nabla \cdot (\alpha \nabla \phi) + \beta \phi - f] \delta \phi \, d\Omega \\ & + \oint_{\Gamma} \alpha \frac{\partial \phi}{\partial n} \delta \phi \, d\Gamma \end{aligned}$$

Equivalent Variational Problem

Since $\phi = p$ on Γ , $\delta\phi = 0$ on Γ . Therefore,

$$\delta F(\phi) = \iint_{\Omega} [-\nabla \cdot (\alpha \nabla \phi) + \beta \phi - f] \delta \phi \, d\Omega$$

Imposing $\delta F(\phi) = 0$, we obtain

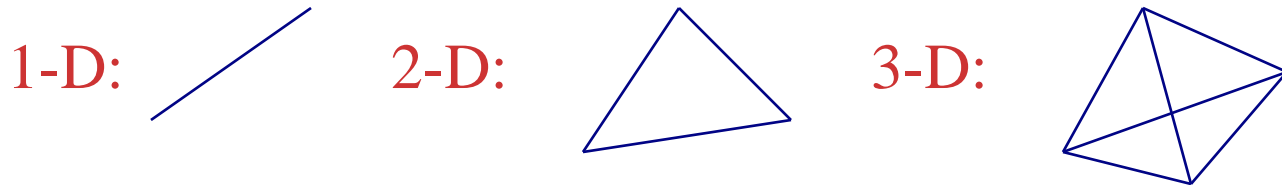
$$\iint_{\Omega} [-\nabla \cdot (\alpha \nabla \phi) + \beta \phi - f] \delta \phi \, d\Omega = 0$$

Because $\delta\phi$ is an arbitrary variation,

$$-\nabla \cdot (\alpha \nabla \phi) + \beta \phi - f = 0$$

Basic FEM Steps

1. Discretization/subdivision of solution domain



2. Selection of interpolation schemes

Linear or higher-order polynomials

3. Formulation of the system of equations

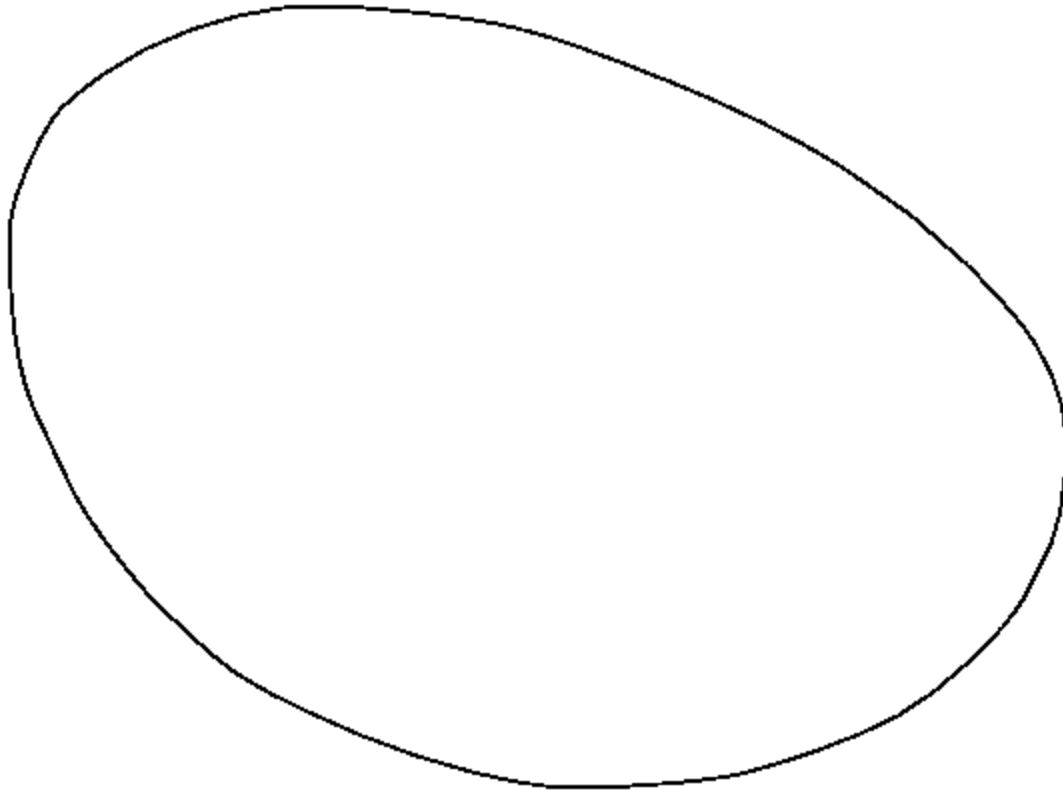
Using either the Ritz or Galerkin method:
Formulate elemental equations and assemble

4. Solution of the system of equations

Using either a direct or iterative method

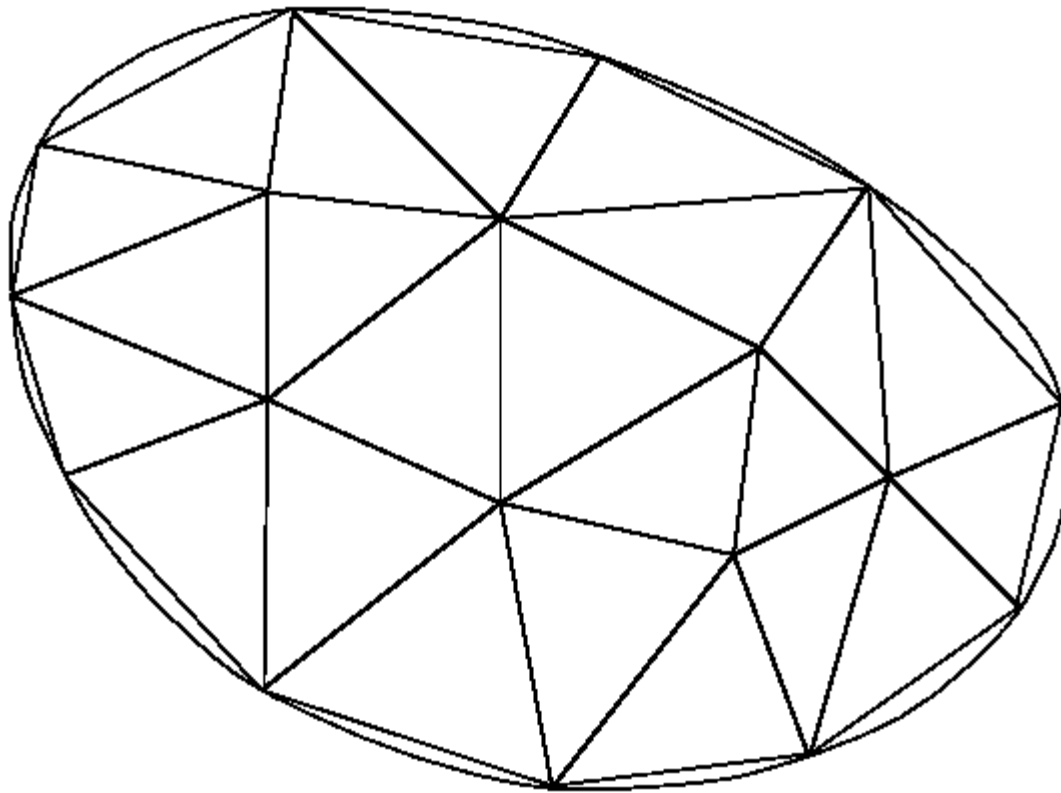
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



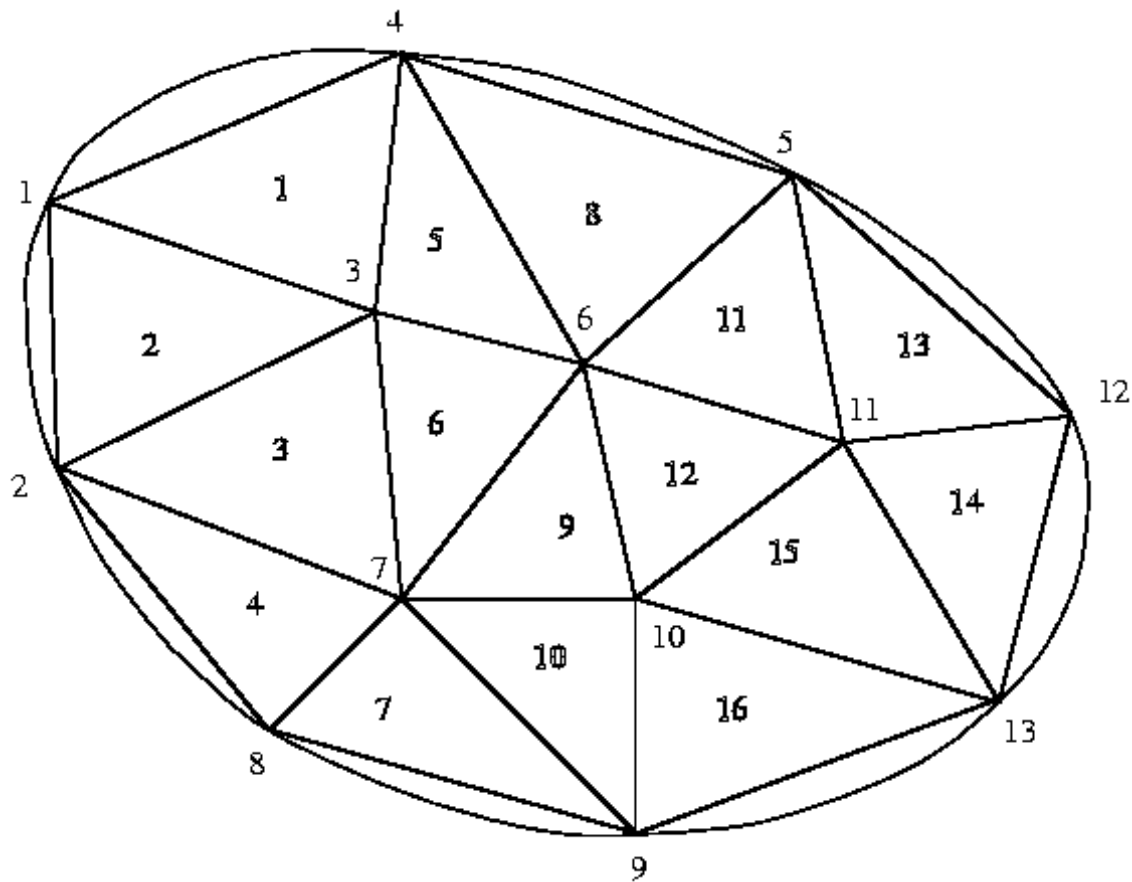
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



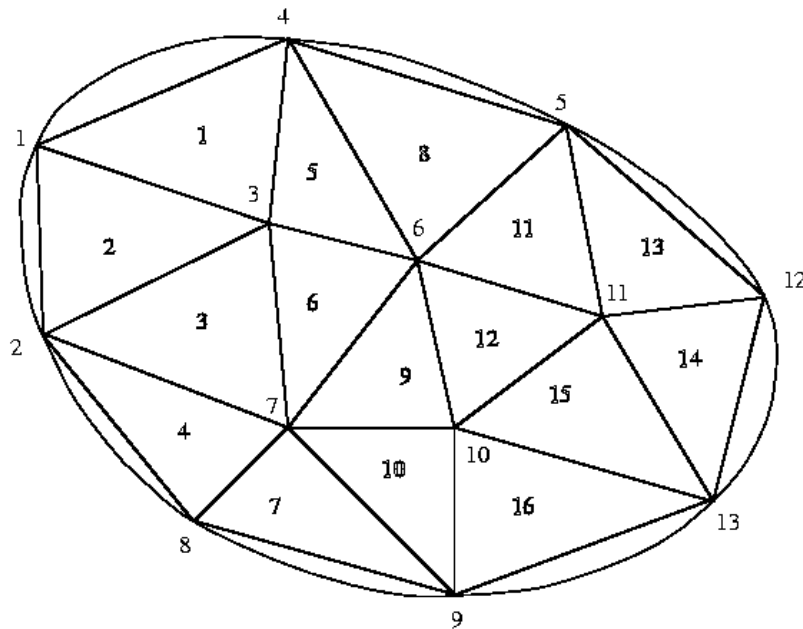
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



FEM Analysis – Domain subdivision

Step 1: Domain Discretization



e	$n(1, e)$	$n(2, e)$	$n(3, e)$
1	1	3	4
2	2	3	1
3	2	7	3
4	8	7	2
5	3	6	4
6	7	6	3
7	8	9	7
8	6	5	4
9	7	10	6
10	9	10	7
.	.	.	.
.	.	.	.
.	.	.	.

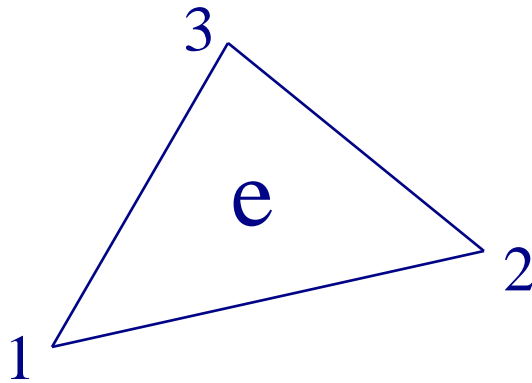
FEM Analysis – Element interpolation

Step 2: Element Interpolation

Assume

$$\phi^e(x, y) = a^e + b^e x + c^e y$$

Apply at the three nodes:



$$\phi_1^e = a^e + b^e x_1^e + c^e y_1^e$$

$$\phi_2^e = a^e + b^e x_2^e + c^e y_2^e$$

$$\phi_3^e = a^e + b^e x_3^e + c^e y_3^e$$

Solve for a^e , b^e , and c^e .

FEM Analysis – Element interpolation

$$\begin{aligned}\phi^e(x, y) &= N_1^e(x, y)\phi_1^e + N_2^e(x, y)\phi_2^e + N_3^e(x, y)\phi_3^e \\ &= \sum_{j=1}^3 N_j^e(x, y)\phi_j^e\end{aligned}$$

$$N_j^e(x, y) = \frac{1}{2\Delta^e} (a_j^e + b_j^e x + c_j^e y) \quad j = 1, 2, 3.$$

$$a_1^e = x_2^e y_3^e - y_2^e x_3^e; \quad b_1^e = y_2^e - y_3^e; \quad c_1^e = x_3^e - x_2^e$$

$$a_2^e = x_3^e y_1^e - y_3^e x_1^e; \quad b_2^e = y_3^e - y_1^e; \quad c_2^e = x_1^e - x_3^e$$

$$a_3^e = x_1^e y_2^e - y_1^e x_2^e; \quad b_3^e = y_1^e - y_2^e; \quad c_3^e = x_2^e - x_1^e$$

Δ^e = area of the eth element

FEM Analysis – Element formulation

Step 3: Formulation of the System of Equations

A. Elemental equations

$$F(\phi) = \sum_{e=1}^M F^e(\phi^e)$$

$$F^e(\phi^e) = \frac{1}{2} \iint_{\Omega^e} \left[\alpha \nabla \phi^e \cdot \nabla \phi^e + \beta (\phi^e)^2 \right] d\Omega \\ - \iint_{\Omega^e} f \phi^e d\Omega$$

FEM Analysis – Element formulation

Elemental functional:

$$F^e = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \phi_i^e \phi_j^e \iint_{\Omega^e} \left(\alpha \nabla N_i^e \cdot \nabla N_j^e + \beta N_i^e N_j^e \right) d\Omega - \sum_{i=1}^3 \phi_i^e \iint_{\Omega^e} f N_i^e d\Omega$$

$$F^e = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \phi_i^e K_{ij}^e \phi_j^e - \sum_{i=1}^3 \phi_i^e b_i^e$$

FEM Analysis – Element formulation

$$K_{ij}^e = \iint_{\Omega^e} \left(\alpha \nabla N_i^e \cdot \nabla N_j^e + \beta N_i^e N_j^e \right) dx dy \quad i, j = 1, 2, 3$$

$$b_i^e = \iint_{\Omega^e} f N_i^e dx dy \quad i = 1, 2, 3.$$

Integration formula:

$$\iint_{\Omega^e} (N_1^e)^l (N_2^e)^m (N_3^e)^n dx dy = \frac{l!m!n!}{(l+m+n+2)!} 2\Delta^e$$

$$K_{ij}^e = \frac{\alpha^e}{4\Delta^e} (b_i^e b_j^e + c_i^e c_j^e) + \frac{\Delta^e}{12} \beta^e (1 + \delta_{ij})$$

$$b_i^e = \frac{\Delta^e}{3} f^e.$$

FEM Analysis – Element formulation

Use matrix notation:

$$F^e = \frac{1}{2} \{\phi^e\}^T [K^e] \{\phi^e\} - \{\phi^e\}^T \{b^e\}$$

$$[K^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix}; \quad \{\phi^e\} = \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix}; \quad \{b^e\} = \begin{Bmatrix} b_1^e \\ b_2^e \\ b_3^e \end{Bmatrix}.$$

FEM Analysis – Assembly

B. Assembly

$$F^e = \frac{1}{2} \{\phi\}^T [\bar{K}^e] \{\phi\} - \{\phi\}^T \{\bar{b}^e\}$$

$$[K^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} \rightarrow [\bar{K}^{(1)}] = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{23}^{(1)} & K_{21}^{(1)} & 0 & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FEM Analysis – Assembly

Carry out the summation:

$$\begin{aligned} F &= \frac{1}{2} \sum_{e=1}^M \{\phi\}^T [\bar{K}^e] \{\phi\} - \sum_{e=1}^M \{\phi\}^T \{\bar{b}^e\} \\ &= \frac{1}{2} \{\phi\}^T \left(\sum_{e=1}^M [\bar{K}^e] \right) \{\phi\} - \{\phi\}^T \left(\sum_{e=1}^M \{\bar{b}^e\} \right) \\ &= \frac{1}{2} \{\phi\}^T [K] \{\phi\} - \{\phi\}^T \{b\} \end{aligned}$$

Apply the stationarity condition:

$$\delta F = 0 \quad \longrightarrow \quad [K] \{\phi\} = \{b\}$$

FEM Analysis – Assembly

How to carry out the summation?

$$[K] = \sum_{e=1}^M [\bar{K}^e], \quad \{b\} = \sum_{e=1}^M \{\bar{b}^e\}$$

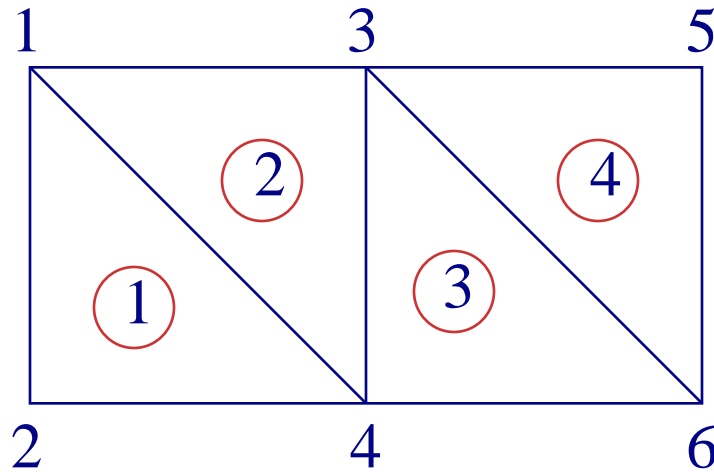
To assemble $[K]$: Add each K_{ij}^e to $K_{n(i,e),n(j,e)}$.

To assemble $\{b\}$: Add each b_i^e to $b_{n(i,e)}$.

FEM Analysis – Assembly

Example:

$$\{\phi\} = \left\{ \begin{array}{l} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{array} \right\}$$



e	$n(1, e)$	$n(2, e)$	$n(3, e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis

1. Start from a null matrix and add in the first element:

$$[K] = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{23}^{(1)} & K_{21}^{(1)} & 0 & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis

2. Add in the second element:

$$[K] = \begin{bmatrix} K_{33}^{(1)} + K_{33}^{(2)} & K_{31}^{(1)} & K_{32}^{(1)} + K_{31}^{(2)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ K_{23}^{(2)} & 0 & K_{22}^{(2)} & K_{21}^{(2)} & 0 & 0 \\ K_{23}^{(1)} + K_{13}^{(2)} & K_{21}^{(1)} & K_{12}^{(2)} & K_{22}^{(1)} + K_{11}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis

3. Add in the third element:

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

$$[K] = \begin{bmatrix} K_{33}^{(1)} + K_{33}^{(2)} & K_{31}^{(1)} & K_{32}^{(2)} & K_{32}^{(1)} + K_{31}^{(2)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ K_{23}^{(2)} & 0 & K_{22}^{(2)} + K_{33}^{(3)} & K_{21}^{(2)} + K_{31}^{(3)} & 0 & K_{32}^{(3)} \\ K_{23}^{(1)} + K_{13}^{(2)} & K_{21}^{(1)} & K_{12}^{(2)} + K_{12}^{(3)} & K_{22}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} & 0 & K_{12}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{23}^{(3)} & K_{21}^{(3)} & 0 & K_{22}^{(3)} \end{bmatrix}$$

FEM Analysis

4. Add in the fourth element:

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

$$[K] = \begin{bmatrix} K_{33}^{(1)} + K_{33}^{(2)} & K_{31}^{(1)} & K_{32}^{(2)} & K_{32}^{(1)} + K_{31}^{(2)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ K_{23}^{(2)} & 0 & K_{22}^{(2)} + K_{33}^{(3)} + K_{33}^{(4)} & K_{21}^{(2)} + K_{31}^{(3)} & K_{32}^{(4)} & K_{32}^{(3)} + K_{31}^{(4)} \\ K_{23}^{(1)} + K_{13}^{(2)} & K_{21}^{(1)} & K_{12}^{(2)} + K_{12}^{(3)} & K_{22}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} & 0 & K_{12}^{(3)} \\ 0 & 0 & K_{23}^{(4)} & 0 & K_{22}^{(4)} & K_{21}^{(4)} \\ 0 & 0 & K_{23}^{(3)} + K_{13}^{(4)} & K_{21}^{(3)} & K_{12}^{(4)} & K_{22}^{(3)} + K_{11}^{(4)} \end{bmatrix}$$

FEM Analysis

5. Follow a similar procedure:

$$\{b\} = \left\{ \begin{array}{c} b_3^{(1)} + b_3^{(2)} \\ b_1^{(1)} \\ b_2^{(2)} + b_3^{(3)} + b_3^{(4)} \\ b_2^{(1)} + b_1^{(2)} + b_1^{(3)} \\ b_2^{(4)} \\ b_2^{(3)} + b_1^{(4)} \end{array} \right\}$$

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis – Apply BC

C. Impose the Dirichlet Boundary Condition:

➤ Approach #1:

To impose $\phi_1 = p_1$, simply set:

$$K_{11} = 1, \quad K_{1i} = 0 \quad \text{for } i = 2, 3, 4, 5, 6, \quad b_1 = p_1$$

To maintain symmetry, set:

$$b_i \leftarrow b_i - K_{i1}p_1, \quad K_{i1} = 0 \quad \text{for } i = 2, 3, 4, 5, 6$$

FEM Analysis – Apply BC

After imposing $\phi_1 = p_1$, $\phi_3 = p_3$, $\phi_5 = p_5$:

$$[K] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & 0 & K_{24} & 0 & K_{26} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & 0 & K_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & K_{62} & 0 & K_{64} & 0 & K_{66} \end{bmatrix}$$

Remains
Symmetric!

$$\{b\} = \left\{ \begin{array}{l} p_1 \\ b_2 - K_{21}p_1 - K_{23}p_3 - K_{25}p_5 \\ p_3 \\ b_4 - K_{41}p_1 - K_{43}p_3 - K_{45}p_5 \\ p_5 \\ b_6 - K_{61}p_1 - K_{63}p_3 - K_{65}p_5 \end{array} \right\}$$

FEM Analysis – Apply BC

Can be made smaller:

$$\begin{bmatrix} K_{22} & K_{24} & K_{26} \\ K_{42} & K_{44} & K_{46} \\ K_{62} & K_{64} & K_{66} \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_4 \\ \phi_6 \end{Bmatrix} = \begin{Bmatrix} b_2 - K_{21}p_1 - K_{23}p_3 - K_{25}p_5 \\ b_4 - K_{41}p_1 - K_{43}p_3 - K_{45}p_5 \\ b_6 - K_{61}p_1 - K_{63}p_3 - K_{65}p_5 \end{Bmatrix}$$

Worthwhile when there are many prescribed nodes.

FEM Analysis – Apply BC

➤ Approach #2 (Simple one):

To impose $\phi_1 = p_1$, simply set:

$$10^{70}\phi_1 + K_{12}\phi_2 + K_{13}\phi_3 + K_{14}\phi_4 + K_{15}\phi_5 + K_{16}\phi_6 = p_1 \times 10^{70}$$

After imposing $\phi_1 = p_1$, $\phi_3 = p_3$, $\phi_5 = p_5$:

$$\begin{bmatrix} 10^{70} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & 10^{70} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & 10^{70} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{Bmatrix} = \begin{Bmatrix} p_1 \times 10^{70} \\ b_2 \\ p_3 \times 10^{70} \\ b_4 \\ p_5 \times 10^{70} \\ b_6 \end{Bmatrix}$$