Optical Dispersion

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A quick review of fiber’s loss characteristics:

Manufacturing of optical fibers
Slide presentation

Linear Propagation in Single Mode Fibers
Impairments:
1. Loss
2. Modal dispersion (only in MMF)
3. GVD (chromatic) dispersion
4. PMD

Origin of optical loss:
(a) Short wavelength loss is due to Rayleigh scattering caused by random density fluctuations leading to refractive index fluctuations. It scales as $\lambda^{-4}$.

(b) Long wavelength absorption due to vibrational resonances. O-H ion has fundamental vibrational absorption peak at ~2.9 um and its overtone causes the peak near 1.4 um. (2.9 um is an important wavelength for laser surgery!)
Fiber Modes
Cause problems similar to multi-path propagation in wireless communication (main problem in wireless since there are no channel nonlinearities)

Parameter, \( V \), is a measure of the number of guided modes and is a very useful parameter,

\[
V = \frac{2\pi}{\lambda} \cdot a \cdot \sqrt{n_1^2 - n_2^2} = \frac{2\pi}{\lambda} a \cdot NA < 2.4 \quad \text{Single mode condition}
\]

\# of modes \( \approx \frac{V^2}{2} \) for large \( V \)

As \( \lambda \downarrow \) more modes
Cutoff wavelength \( \equiv \) smallest \( \lambda \) that satisfy single mode condition.

As \( a \uparrow \) more modes
\( \Rightarrow \) tradeoff between dispersion impairments and light coupling efficiency.

As \( \Delta = \frac{n_1 - n_2}{n_1} \uparrow \) more modes
\( \Rightarrow \) to have single mode operation, we want weakly confined waveguide guide

Result: significant portion of optical power propagates in cladding

Typical single mode fibers, \( a = 7 \mu m, \Delta = 0.001 \) at \( \lambda = 1.55 \mu m \)
Typical MMF; \( a = 50 \mu m, \Delta = 0.005 \)

MMF support hundreds of modes. Mode spacing decreases with increasing \( a/\lambda \), becoming a continuum for highly multimode waveguides.

**Polarization**

- Fiber is a cylindrical waveguide

- General E field
  \[
  \vec{E}(r,t) = \vec{E}_x + \vec{E}_y + \vec{E}_z
  \]
  \( \vec{E}_i \) can be a function of \( x, y, z \)

The lowest order mode is the HE_{11} (hybrid electric) mode
This mode is doubly degenerate.

-2 linearly independent hybrid solutions of the wave equation exist
  for fundamental mode:
(1) $\vec{E}_x = 0$, $\vec{E}_y, \vec{E}_z \neq 0$
(2) $\vec{E}_y = 0$, $\vec{E}_x, \vec{E}_z \neq 0$

- For fundamental mode, $\vec{E}_z \ll \vec{E}_x, \vec{E}_y$

⇒ So effectively, we have a transversal fundamental mode, each solution looks like

$$\vec{E}(r, \omega) = 2\pi \cdot J(x, y) \cdot e^{i\beta(\omega)z}$$

$J \equiv$ Bessel function of the first kind of order zero;
$J(x, y) = J(\rho)$

Radius $\rho = \sqrt{x^2 + y^2}$, cylindrical symmetry

- There are then 2 solutions - both transversal,
  $E_x, E_y$ are both linearly polarized

Any linear combination is also a solution.
Both modes have same propagation constant (circular symm)
⇒ still single mode. No dispersion.
-in practice, fiber core is not perfectly circular => Birefringence

\[ E_x \text{ and } E_y \text{ have different propagation constants} \]

\[
\beta = \frac{n \cdot 2\pi}{\lambda} = n \cdot k \quad n_2 k < \beta < n_1 k \quad \text{(insight: propagation constant for mode is a weighted average of the constant for core and cladding)}
\]

\[
\beta_x > \beta_y \quad \text{because mode overlap with core is larger in along x than y}
\]

-difference in \( \beta \) for \( E_x \text{ and } E_y \) \( \rightarrow \) PMD: Polarization Mode Dispersion

**PMD**

-Due to birefringence of fiber

source of birefringence

- Geometrical irregularities (core is not circular)
- Internal stress, external bending, twisting, pinching
- 1% deviation from circularity will have noticeable effect at >10G b/s communication

\[
\Delta \tau_{PMD} = \frac{L}{v_{gx}} - \frac{L}{v_{gy}} \quad \text{Differential Group Delay (DGD)}
\]

-in contrast to chromatic dispersion, PMD varies with time

-temperature along fiber changes

requires dynamic compensation

-PMD parameter

\[ \langle \Delta \tau_{PMD} \rangle = D_{PMD} \sqrt{L} \]

unit of \( D_{PMD} \) \( \text{ps} / \sqrt{\text{km}} \); random walk process, proportional to \( L^{1/2} \)

it’s an average value
Traditional value in modern fibers $0.1 \ ps / \sqrt{\text{km}}$ (less for modern fibers)
-depends on fiber and also installation method
Aerial fibers show larger $D_{PMD}$
Due to sudden changes in temperature and movement due to wind

Example:
Assume $D = 1 \text{ps/km}^{1/2}$, and $L = 100\text{km}$
Pulse broadening, $\text{DGD} = 10\text{ps}$
Take a $\text{B}=10\text{Gb/s}$ system,
Symbol period, $1/B = 100\text{ps}$ for NRZ and $50\text{ps}$ for RZ. Here PMD is not a problem
Now take a $\text{B}=40\text{Gb/s}$ system,
Symbol period = $25\text{ps}$ for NRZ and $12.5\text{ps}$ for RZ. Here PMD is a problem
Case study: for “100G” standard ($28\text{G} \times 4 = 112 \text{Gbps}$) PMD DGD spec is $10\text{ps}$ mean and $30\text{ps}$ peak-to-peak.

Various researchers and companies have developed either optical or electrical (equalization) solutions

PMD Specification:

- First order PMD is a vector (with magnitude equal to DGD) and direction as one of the two principle states of polarization (PSP).
- In optical links, both DGD and PSP are optical frequency dependent, hence second order PMD, which is a derivative of first order PMD with frequency, is nonzero.
- For 100G (OTU4) transport standard, The first order PMD specification is DGD (average) = $10\text{-ps}$ average, DGD (peak) = $30\text{ps}$.
- PMD is frequency dependent. For broad optical bandwidths and long links, second order PMD also becomes important and can be specified in units of $\text{ps/nm}$. Its magnitude and implications are currently being studied for next generation optical links.
- Good reference on PMD specification:

**Effect of PMD on analog links (e.g. CATV or antenna remoting links)**

**Approach 1:**
\[ E_x = \sqrt{P_o} e^{i\varphi_x} \]
\[ E_y = \sqrt{P_o} e^{i\varphi_y} \]
\[ P_{out} = |E_x + E_y|^2 = \left( \sqrt{P_o} e^{i\varphi_x} + \sqrt{P_o} e^{i\varphi_y} \right) \left( \sqrt{P_o} e^{-i\varphi_x} + \sqrt{P_o} e^{-i\varphi_y} \right) = 2P_o + P_o \left[ e^{i(\varphi_y - \varphi_x)} + e^{-i(\varphi_y - \varphi_x)} \right] = 2P_o \left[ 1 + \cos(\varphi_y - \varphi_x) \right] = 2P_o \cdot \cos^2 \left( \frac{\varphi_y - \varphi_x}{2} \right) \]
\[ |H(\omega_m)|^2 \propto \cos^2 \left( \frac{\varphi_y - \varphi_x}{2} \right) \]
\[ \Delta \varphi = \varphi_y - \varphi_x = \Delta \beta \cdot l = \frac{\omega_m}{c} \Delta n \cdot l = \frac{\omega_m}{c} |n_x - n_y| \cdot l \]
\[ |H(\omega_m)|^2 \propto \cos^2 \left( \frac{\omega_m}{2c} |n_x - n_y| \cdot l \right) \]

**Approach 2 (Alternative):**

Let the carrier wave be \[ E_0 = \frac{A}{\sqrt{2}} e^{j\omega_0 t} \left( \hat{x} + \hat{y} \right) \], linearly polarized at an angle 45° from the X-axis.

Let the modulation frequency be \( \omega_m \)

The two side bands at the input can be represented as:

\[ E_{1in}^{in} = \frac{a}{\sqrt{2}} e^{j(\omega_0 - \omega_m) t} \left( \hat{x} + \hat{y} \right) \]
\[ E_{2in}^{in} = \frac{a}{\sqrt{2}} e^{j(\omega_0 + \omega_m) t} \left( \hat{x} + \hat{y} \right) \]

The carrier and two side bands after propagating through the fiber can be represented as:

\[ E_0 = \frac{A}{\sqrt{2}} e^{j\omega_0 t} \left( x e^{-j\delta_0 x} + y e^{-j\delta_0 y} \right) \]
\[ E_1^{\text{out}} = \frac{a}{\sqrt{2}} e^{j(\sigma_0 - \sigma_m) t} \left( x e^{-j\delta_1^x} + y e^{-j\delta_1^y} \right) \]

\[ E_2^{\text{out}} = \frac{a}{\sqrt{2}} e^{j(\sigma_0 + \sigma_m) t} \left( x e^{-j\delta_2^x} + y e^{-j\delta_2^y} \right) \]

\[ \delta_0^x = \frac{2m_1 L}{\lambda_0}, \delta_0^y = \frac{2m_1 L}{\lambda_0} \]

\[ \delta_1^x = \frac{2m_1 L}{\lambda_1}, \delta_1^y = \frac{2m_1 L}{\lambda_1} \]

\[ \delta_2^x = \frac{2m_1 L}{\lambda_2}, \delta_2^y = \frac{2m_1 L}{\lambda_2} \]

where,

\[ E_0^{\text{out}} + E_1^{\text{out}} + E_2^{\text{out}} = \]

\[ \frac{A}{\sqrt{2}} e^{j(\sigma_0) t} e^{-j\delta_0^x} \left( x + y e^{-j(\delta_0^x - \delta_0^y)} \right) + \frac{a}{\sqrt{2}} e^{j(\sigma_0 - \sigma_m) t} e^{-j\delta_1^x} \left( x + y e^{-j(\delta_1^x - \delta_1^y)} \right) \]

\[ \frac{a}{\sqrt{2}} e^{j(\sigma_0 + \sigma_m) t} e^{-j\delta_2^x} \left( x + y e^{-j(\delta_2^x - \delta_2^y)} \right) \]
\[ P_{\text{opt}} = \left( E_0^{\text{out}} + E_1^{\text{out}} + E_2^{\text{out}} \right) \left( E_0^{\text{out}} + E_1^{\text{out}} + E_2^{\text{out}} \right)^* \]

\[ = \left( \frac{A}{\sqrt{2}} e^{j(\sigma_0)t} . e^{-j\delta_0^t} \left( x + y e^{j(\delta_0^x - \delta_0^t)} \right) + \frac{a}{\sqrt{2}} e^{j(\sigma_0 - \sigma_m)t} . e^{-j\delta_0^t} \left( x + y e^{j(\delta_0^x - \delta_0^t)} \right) \right) \]

\[ + \left( \frac{A}{\sqrt{2}} e^{-j(\sigma_0)t} . e^{+j\delta_0^t} \left( x + y e^{-j(\delta_0^x - \delta_0^t)} \right) + \frac{a}{\sqrt{2}} e^{-j(\sigma_0 - \sigma_m)t} . e^{+j\delta_0^t} \left( x + y e^{-j(\delta_0^x - \delta_0^t)} \right) \right) \]
Transfer function is given as:

\[
T(\omega_m) \propto \cos^2 \left( \frac{\omega_m(n_y-n_x)L}{c} \right)
\]
Before we talk about Group Velocity Dispersion (GVD) also called chromatic dispersion (CD) let’s review different definitions of velocity:

**Phase Velocity**

For a monochromatic (single frequency) lightwave in fiber, we have the following solution;

\[ E(\rho, \omega) = 2\pi \cdot J(\rho) \cdot e^{i\beta(\omega)z} \]

E vector is in \( \hat{x} \) or \( \hat{y} \) for the doubly degenerate fundamental mode

In time domain, the magnitude of E is:

\[ |E(\rho, t)| = J(\rho)\cos(\omega_0 t - \beta z) \]

**Phase velocity is how fast the peak travels**

Let \( V_p \) be the propagation velocity, then after \( t = t_1 \) the peak has traveled by distance \( z \),

phase change per unit length of travel = \( \beta \)

phase change per unit time of travel = \( \omega_0 \)

After \( t = t_1 \);

\[ z = V_p \cdot t_1 \]

\[ \omega_0 \cdot t_1 = \beta \cdot V_p \cdot t_1 \Rightarrow V_p = \frac{\omega_0}{\beta} \]

**Group Velocity**

- Monochromatic wave cannot carry any information. In communication, we instead send pulses, or a modulated monochromatic carrier wave. The signal then consist of many frequencies.

- let’s take a simple pulse consisting of 2 frequency tones, closely spaced around a center frequency,

\[ \omega_1 = \omega_0 + \Delta \omega \]

\[ \omega_2 = \omega_0 - \Delta \omega \]

This corresponds to a simple AC modulation where we have created an upper and lower sideband of the carrier (monochromatic) wave.
if $\Delta \omega$ is small $\Rightarrow \beta(\omega)$ can be approximated as a linear function (Taylor expansion)

$$\beta(\omega_0 \pm \Delta \omega) \approx \beta_0 \pm \beta_1 \Delta \omega$$

Power is proportional to magnitude square of E field:

$$|E(r,t)|^2 = E(r,t) \cdot E(r,t)^* = |E_1(r,t) + E_2(r,t)| \cdot |E_1(r,t) + E_2(r,t)^*|^* = DC + AC$$

The AC term will be proportional to

$$2J(\rho)\cos(\Delta \omega \cdot t - \beta_1 \Delta \omega z) \cdot \cos(\omega_0 t - \beta_0 z)$$

where we used: $\cos A + \cos B = 2 \cos(\frac{A+B}{2}) \cdot \cos(\frac{A-B}{2})$

-For the carrier:

$$v_\rho = \frac{\omega_0}{\beta_0}$$

But envelope travels at:

$$v_g = \frac{1}{\beta_1}$$

called the group velocity, where

$$\beta_1 \equiv \frac{d\beta}{d\omega} \bigg|_{\omega_0}$$

Information is carried by the envelope

If we have more than 2-tones, envelope will not be a sinusoid but a shaped pulse

**Group Velocity (Chromatic) Dispersion (GVD)

A pulse or a modulated CW wave has finite frequency spread. Due to chromatic dispersion, different frequency constituents of the pulse travel with different phase velocities causing the pulse to spread.
GVD is caused by two types of dispersion:

1. Waveguide Dispersion
2. Material Dispersion

**Contribution 1: Waveguide Dispersion**

- Due to redistribution of light within the waveguide

As $\lambda \uparrow$ more light sits in the cladding, where $n$ is lower, hence the effective mode index is a function of wavelength (or frequency) $\Rightarrow n(\lambda)$

Mode index decreases with increasing wavelength
Waveguide dispersion is engineered through doping concentration and waveguide core/cladding sizes

**Contribution 2: Material Dispersion**

- All material have $n$ that’s a function of $\lambda$ “material dispersion”

  **Material Dispersion**

- Comes from frequency dependence of $n(\omega)$

- Medium absorbs radiation through oscillations of bound electrons. Characteristic resonant frequencies cause material dispersion

- Far from resonance, $n(\omega)$ is described by the Sellmeier equation: (Harmonic Oscillator model for $n$)

  $$n^2(\omega) = 1 + \sum_{j=1}^{m} \frac{B_j}{\omega_j^2 - \omega^2}$$

  Sum over all resonances (Cardona book, section 6.4) – valid only at frequencies far from resonance

  $B_j$ and $\omega_j$ are the strength and frequency of the $j^{th}$ resonance.

In fiber, these parameters are obtained by fitting the experimentally measured dispersion curves to the equation with $m=3$. Phenomenological approach.
Material dispersion is not easily manipulated in a given material => what is known as “dispersion engineering” is mostly an exercise in waveguide design

**Contribution 3: Optical nonlinearities cause GVD**
GVD also has a contribution from nonlinear effects – refractive index becomes a function of optical intensity: Self Phase Modulation (SPM) – will be covered when discussing nonlinear optical effects

**GVD Parameters**
Combined effects of waveguide and material dispersion is called chromatic dispersion.

- cause the frequency dependence of $n$ and $\beta$

$$\beta(\omega) = n(\omega) \frac{\omega}{c} = \beta_o + \beta_1(\omega - \omega_o) + \frac{1}{2} \beta_2(\omega - \omega_o)^2 + ...$$

$$\beta_m = \left(\frac{d^m \beta}{d\omega^m}\right)_{\omega=\omega_o} \quad m = 0,1,2,...$$

$$\beta_1 = \frac{d\beta}{d\omega} = \frac{1}{v_g} = \frac{n_g}{c} = \frac{1}{c} (n + \omega \frac{dn}{d\omega})$$

$$n_g \equiv (n + \omega \frac{dn}{d\omega})$$

$$\beta_2 = \frac{d^2 \beta}{d\omega^2} = \frac{1}{c} (2 \frac{dn}{d\omega} + \omega \frac{d^2 n}{d\omega^2})$$

$$\frac{dv_g}{d\beta} = \frac{d^2 \omega}{d\beta^2} \equiv \beta_2 \ ;$$

In addition to $\beta_2$, another parameter used to describe chromatic dispersion is the $D$ parameter,

$$\text{Differential delay} = \frac{\Delta \tau}{\Delta \lambda} = D(\lambda)$$

Conversion between $D$ and $\beta_2$:

$$D = \frac{-2 \cdot \pi \cdot c}{\lambda^2} \beta_2$$
when D is in ps/km.nm and beta_2 is in ps^2/km

then \( \beta_2 = -1.3[nm \cdot ps] \cdot D \) at 1550nm

Example: for SMF-28 fiber, D = 16 ps/nm.km, hence beta_2 \( \sim -21 \) ps^2/km

Note: D and beta_2 have opposite signs

Dispersion terminology:

“Anomalous Dispersion”: region of negative beta_2 (positive D)

“Normal Dispersion”: region of positive beta_2 (negative D)

Dispersion Slope:

\( S(\lambda) = dD / d\lambda \) at \( \lambda_o \); at zero dispersion wavelength

\( S_o \sim 0.092 \) ps/(nm^2.km) at zero dispersion wavelength

Figure 2.11: Typical wavelength dependence of the dispersion parameter D for standard, dispersion-shifted, and dispersion-flattened fibers.
For standard single mode fiber (SMF28) at 1550nm:

\[ D = 16 \text{ ps/km.nm} \]

\[ \beta_2 = -21 \text{ ps}^2/\text{km} \text{ (anomalous dispersion regime)} \]

\[ \beta_3 = 0.1 \text{ ps}^3/\text{km} \]
GVD Model
Dispersion depends on wavelength and on fiber type (waveguide design). The international telecommunication Standards Committee ((ITU-T Recommendation G.650) has defined the standard model for fiber dispersion.

To calculate the dispersion for non DSF, the Standard calls for fitting the measured group delay per unit length to a Sellmeier equation (in terms of $\lambda$) of type:

$$\tau(\lambda) = \tau_o + \frac{S_o}{8} \left(\frac{\lambda - \lambda_o^2}{\lambda}\right)^2$$

Where $\tau_o$ is the relative delay minimum (with respect to zero dispersion) at zero dispersion wavelength, $\lambda_o$.
$S_o$ is the value of the dispersion slope $S(\lambda) = dD/d\lambda$ at $\lambda_o$. Units: ps/nm^2.km).

For DSF, a quadratic expression is used:

$$\tau = \tau_o + \frac{S_o}{2} (\lambda - \lambda_o)^2$$

which results in dispersion expression:

$$D(\lambda) = \frac{d\tau}{d\lambda} = (\lambda - \lambda_o) S_o$$

$S_o \sim 0.092$ ps/(nm^2.km) for standard non-DSF
$\sim 0.07$ for DSF
Impact of GVD on Propagation of Gaussian Pulses

In general, if there are many tones centered around $\omega_0$ and we have “narrow spectral width.” Narrow spectrum also means “slowly varying envelope approximation”.

Then,

$$|E(z, t)| = J(\rho) \text{Re}[A(z, t)e^{-i(\omega_0t - \beta_0z)}]$$

Transverse profile envelope

If $\frac{d\beta_1}{d\omega} = \frac{d^2\beta}{d\omega^2} = \beta_2 = 0 \Rightarrow$ pulse propagation with velocity $\frac{1}{\beta_1}$

If $\beta_2 \neq 0$ different $\omega$’s have different velocity – we have GVD.

$\Rightarrow$ different parts of wave propagate at different velocity $\Rightarrow$ Pulse broadending $\Rightarrow$ Inter Symbol Interference (ISI)

This is specially problematic in two cases:
1. chirped pulses – initially chirped or chirped due to SPM
2. spectrally broad (temporally narrow) pulses

Chirped Gaussian pulses

Envelope assumed to be Gaussian
Chirped: freq. or $\lambda$ changes with time
Sources of chirp: Light emitted by directly modulated lasers is chirped
Also NL effects can chirp an un-chirped pulse.

at $z=0$

$$A(\omega, t) = A_0e^{-\frac{1+ik}{2\left(\frac{t}{T_0}\right)^2}}$$

input pulse

$\hat{k} \equiv$ chirp factor

$$= A_0e^{-\frac{1}{2}\left(\frac{t}{T_0}\right)^2} \cdot e^{-\frac{ik}{2}\left(\frac{t}{T_0}\right)^2} \quad \phi \propto t^2 \Rightarrow \text{freq} \propto t$$
Propagation of Chirped Optical Pulses in Linear Media

To understand the propagation of chirped pulses in linear media, we can simplify the NLSE:

\[
\frac{\partial A}{\partial Z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} = 0 \quad \beta_2 = \frac{\partial^2 \beta}{\partial \omega^2}
\] (2)

*can derive the equation from Eqn 2.3.22 of Agrawal NLFO 2nd Edition.

Note: the 1st and 2nd terms are also imaginary due to the derivates.

First term describes the change of A with propagation distance z. The second term is the constant delay due to propagation at velocity 1/\(\beta_1\). In a frame of reference moving with the pulse, this term is zero. 3rd term is the GVD

So nonlinear optical effects are ignored for now.

Pulse shape after propagating over z can be found by solving (2) with (1) as initial condition. Solution is:

\[
A(z,t) = \frac{A_0 T_0}{\sqrt{T_0^2 - i \beta_2 z (1 + ik)}} \exp \left( - \frac{(1 + ik)(t - \beta_1z)^2}{2(T_0^2 - i \beta_2 z (1 + ik))} \right)
\]

Pulse width after propagation through length, z, is:

\[
T_z^2 \rightarrow T_0^2 - i \beta_2 z (1 + ik)
\]

pulse width increases by ratio:

\[
\frac{T_z}{T_0} = \sqrt{\left(1 + \frac{k \beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0}\right)^2}
\]

Broadening due to chirp

Broadening due to finite bandwidth

If \(k \beta_2 > 0\), pulse width increases monotonically with z \(\Rightarrow\) leads to intersymbol interference (ISI)

Also note that the amplitude has decreased by

\[
\frac{A}{A_0} = \frac{T_0}{\sqrt{T_0^2 - i \beta_2 z (1 + ik)}}
\]

“Energy must be conserved”
Note, $\beta_2 \sim -20 \text{ ps}^2/\text{km}$ at 1550nm for standard SMF (anomalous dispersion regime)

**Dispersion Length**, $L_D = \frac{T_o^2}{|\beta_2|}$
Eye Diagrams

- time domain measurement
- Data seen on the scope is random but scope can trigger off of it since the master data clock is provided to it
- At any bit interval, the data can be either 0 or 1
- It’s like overlapping lots of different patterns. For example, consider different permutations of a 3 bit data stream

When many different patterns appear on the scope, an eye diagram is formed:
Eye Masks used to measure number of errors. For example, for 40G (OC-768) the standard mask is: STM256/OC768
“Fig. 12. 40-Gbps NRZ optical waveform of the asymmetric QW EAM. The standard STM256/OC768 mask is aligned to the mark/space average level of the optical waveform and the center of the time slot. “

Y. Miyazaki et al. IEEE JOURNAL OF QUANTUM ELECTRONICS, VOL. 39, NO. 6, JUNE 2003
Exercise: Small Signal Bandwidth of the Single Mode Fiber

1. Obtain the small signal bandwidth of a single mode fiber – closed form expression
   Hint: Consider only GVD. Ignore optical loss and nonlinear optical effects.
   Start with a double sideband modulated optical carrier of the form:

   \[ E(0) = A \exp(i \omega_o t) + a \exp(i(\omega_o + \omega_m) t) + a \exp(i(\omega_o - \omega_m) t) \]
   where \( a \ll A \) (small signal modulation)

2. Plot the small signal transfer function for the intensity (power) of of the optical field vs. modulation frequency, for a SMF-28 fiber with 120 km length.

3. Plot the power spectral density for 10 Gbps non-return to zero (NRZ) data and compare it with the bandwidth of your fiber

3. Describe two methods for measuring GVD.
Solution to Exercise: Small Signal Bandwidth of the Single Mode Fiber

Compare this analysis to that of Agrawal FO System Design, section 2.4.4 Agrawal’s analysis applies to the case where the source is not monochromatic but has a finite bandwidth.

Remember, the physical interpretation of $\beta$ is phase per unit length. After propagating a length $z$, the phase of the carrier is changed by $\beta z$.

Recall that group delay is:

$$\tau(\omega) = \frac{d\phi}{d\omega}$$

Recall $\beta$:

$$\beta(\omega) = \beta_o + \beta_1(\omega - \omega_o) + \frac{1}{2} \beta_2 (\omega - \omega_o)^2 + \ldots$$

The second term is simply the (constant) propagation delay and the third term describes the difference in propagation delay for different frequency components, in other words, Chromatic Dispersion, or, Group Velocity Dispersion (GVD)

Thus if two frequency components are separated by $\Delta\omega$, then after traveling over a unit length, they arrive with phase difference:

$$\phi = \frac{1}{2} \beta_2 \cdot (\Delta\omega)^2$$

dispersion induced phase shift

Conclusion: In general, a wave propagating through the fiber suffers a phase shift such that at after a length $z$,

$$E(z, \omega) = E(0, \omega) \exp \left( i \frac{\beta_2 \omega^2 z}{2} \right)$$

where $\omega$ is measured from the zero dispersion frequency.

Consider a monochromatic cw electric field modulated at frequency $\omega_m$,

$$E(0) = A \exp(i\omega_o t) + a \exp(i(\omega_o + \omega_m) t) + a \exp(i(\omega_o - \omega_m) t)$$

where $a \ll A$ (small signal modulation)

Assuming $\omega_o$ is at zero dispersion frequency, after propagating through a length $z$ of fiber,
\[ E(z) = A \exp(i\omega_0 t) + a \cdot \exp(i(\omega_0 + \omega_m) t) \cdot \exp \left( \frac{i \beta_2 \omega_m^2 z}{2} \right) + a \cdot \exp(i(\omega_0 - \omega_m) t) \cdot \exp \left( \frac{-i \beta_2 \omega_m^2 z}{2} \right) \]

The photocurrent is proportional to the light intensity which is proportional to \( E E^* \). Ignoring terms proportional to \( a^2 \),

\[ E E^* = A^2 + Aa \exp(i \omega_m t) \cdot \exp \left( \frac{i \beta_2 \omega_m^2 z}{2} \right) + Aa \exp(-i \omega_m t) \cdot \exp \left( \frac{-i \beta_2 \omega_m^2 z}{2} \right) + Aa \exp(-i \omega_m t) \cdot \exp \left( \frac{i \beta_2 \omega_m^2 z}{2} \right) \]

\[ + Aa \exp(i \omega_m t) \cdot \exp \left( \frac{-i \beta_2 \omega_m^2 z}{2} \right) + O(a^2) \]

\[ = A^2 + 2 \left[ Aa \cos(\omega_m t) \cdot \exp \left( \frac{i \beta_2 \omega_m^2 z}{2} \right) + Aa \cos(\omega_m t) \cdot \exp \left( \frac{-i \beta_2 \omega_m^2 z}{2} \right) \right] \]

\[ = A^2 + 4Aa \cos(\omega_m t) \cdot \cos \left( \frac{\beta_2 \omega_m^2 z}{2} \right) \]
Insight: in RF domain, fiber acts as a LPF. In optical domain, it's a passband filter.

Case study: For 120km of fiber, \( f_{3dB} \approx 3.5 \) GHz

Power Spectral Density for NRZ DATA:

For 10 Gbps, first null is at 10 GHz

Lowest frequency needed to reproduce highest frequency, 0101..., pattern is a sinusoide at 5GHz. So a bandwidth starting at very low frequency (needed for 00001111... type patterns) is from a few MHz to 5 GHz.

To have some flat top and bottom, to allow some tolerance for the jitter of sample clock, we want more than this.
But we don’t want too much bandwidth because it introduces too much noise into the signal.

A rule of thumb is to have 0.7B. It represents a compromise between the need for tolerance with respect to clock jitter and the need to minimize noise.

**Measurement of Chromatic Dispersion**
- Need to measure group delay as a function of wavelength
- Output of a tunable laser is intensity modulated using RF tone
- At Rx, the RF phase is measured as a function of RF frequency
- The slope is the group delay
- Measurement is repeated for a different wavelength
- Group delay vs. wavelength is plotted
- $D$ parameter is plotted as the derivative of previous plot

![Diagram of Chromatic Dispersion Measurement](image)

*Figure 1.14* (a) Chromatic dispersion measurement of two-port optical devices. (b) Relative group delay versus wavelength. (c) Dispersion parameter versus wavelength.
Methods to combat chromatic dispersion
See also Chapter 9 of Agrawal, FO System Design (2ed).

2. Optical
   - Dispersion compensating fiber (DCF)
   - $D_{SMF} = 17\text{ps/km.nm}$
   - $D_{DCF} = -100\text{ps/km.nm}$
   - Can compensate for 3rd order dispersion (Agrawal section 3.4.3)
   - Fiber Bragg grating

3. Electrical
   - Equalization filter
   - Soft decision
   - Maximum Likelihood Decision (MLD)

Demonstration of dispersion compensation using VPI simulation tool.
But first, we need to understand eye diagrams.
Dispersive Fourier Transform:

Let’s mathematically examine the role of GVD on the Fourier components of a hypothetical optical field.

Recall:

\[ \beta(\omega) = n(\omega) \frac{\omega}{c} = \beta_0 + \beta_1 (\omega - \omega_0) + \frac{1}{2} \beta_2 (\omega - \omega_0)^2 + \ldots \]

After propagation over length, \( z \), the phase each spectral component of the field is changed by

\[ e^{-i\beta_z} = e^{\frac{-i\beta_2}{2} \cdot \omega^2 \cdot z / 2} \]

Let’s now look at what a dispersed waveform will look like in time. That’s what we would see on an oscilloscope.

So we need to perform a Fourier transform (FT) on the spectral domain waveform.

The FT relation for a dispersed waveform is:

\[ E(t) = \int \tilde{E}(\omega) \cdot e^{-i\beta_2 \cdot \omega^2 \cdot z / 2} \cdot e^{i\omega \cdot t} d\omega, \]

where \( E(t) \) is the temporal profile of the dispersed E field, \( \tilde{E}(\omega) \) is the Fourier transform, or equivalently the spectrum of the field, and \( \beta_z z \) is the GVD (group velocity dispersion) produced by a linearly dispersive medium of length \( z \).

We can manipulate the argument of the exponent by adding and subtracting the same term from it:
The last term is not a function of frequency and can be taken out of the Fourier integral.

$$E(t) = e^{\frac{it^2}{2\beta z}} \int \tilde{E}(\omega) e^{-\frac{i\beta z}{2} \left( \omega - \frac{t}{\beta z} \right)^2} d\omega$$

We are interested in the effect of GVD on the measure photocurrent as a function of time. This is what we will see on an oscilloscope.

The photodetector responds to optical power, which is the intensity multiplied mode area. Therefore,

$$\text{Photocurrent} \propto \text{Optical power} \propto E(t) \cdot E^*(t) = |E(t)|^2$$

$$|E(t)| = \int |\tilde{E}(\omega)| e^{-\frac{i\beta z}{2} \left( \omega - \frac{t}{\beta z} \right)^2} d\omega$$

If the GVD, i.e. $\beta z$ is large, then only $\omega_{opt} = \frac{t}{\beta z}$ contributes to the integral. For other frequencies the argument oscillates rapidly and averages out to zero. So for every $t$, only
one frequency survives and that frequency depends linearly on t. In other words, there is a one-to-one correspondence between time and frequency.

We see that *optical frequency becomes linearly proportional to time*

We have achieved *frequency-time mapping*: \( (\beta_2 z) \cdot \omega = t \). We call this Dispersive Fourier Transform (DFT).

**Condition for validity of frequency-time mapping:**

The mapping occurs only when the waveform is sufficiently dispersed, i.e. \( \beta_2 z \) is large.

In other words, only when the Stationary Phase Approximation is valid.

\[
|E(t)| = \left| \int \widetilde{E} (\omega) e^{i\omega t} d\omega \right| = |\widetilde{E}(\omega)|
\]

**Relation to bulk optics:**

This is the time domain version of the well known stationary phase approximation in conventional diffractive optics.

Here we see that the same thing happens during time domain. This is a consequence of the *equivalence between paraxial diffraction and temporal dispersion*.

The profile of the dispersed signal measured on an oscilloscope assumes the shape of its spectrum.
The equivalent in the space-domain is the Fraunhofer diffraction from classical optics, which states that the far-field diffraction pattern of an aperture is given by the Fourier transform of its transmittance function.

In conventional optics, a prism or a diffraction grating performs a Fourier transform. Therefore, any optical element with GVD performs the same function as a prism or a diffraction grating.

But with a major difference:
With GVD, the spectrum is mapped into time, whereas a prism or grating maps it into space.

So we have just discovered a new type of spectrum analyzer that can be used to identify the chemicals and biological samples by measuring their chemical signature.

Resolution bandwidth of the dispersive Fourier transform (DispFT) spectrometer

After completing the square, the phase term in the integrand became:

\[
\frac{-i\beta z}{2} \left( \frac{\omega - t}{\beta z} \right)^2 e^{i\varphi} = \cos \varphi + i \sin \varphi
\]
The horizontal axis here is frequency normalized to optimum frequency 
\( \omega_{opt} = t / \beta_2 z \) and centered at zero:

\[
\bar{\omega} = \frac{\omega}{\omega_{opt}} - 1
\]

Since rapid oscillations between positive and negative values take place as the frequency varies relative, only the central part of cos and sin survives in the integral. The width of this central part in Figure can be obtained by defining the width to be the range between the first zeros of the real part of the exponential, or the two central peaks of its imaginary part which occur at +/- \( \pi / 2 \).

\[
\beta_2 z \left( \omega_{opt} \pm \delta \omega - \frac{t}{\beta_2 z} \right)^2 = \pm \frac{\pi}{2}
\]

\( \omega_{opt} = t / \beta_2 z \)

\[
\beta_2 z (\pm \delta \omega)^2 = \pm \frac{\pi}{2}
\]

\[
\pm \delta \omega = \pm \sqrt{\frac{\pi}{\beta_2 z}}
\]

\[
|\delta \omega| = 2 \sqrt{\frac{\pi}{\beta_2 z}}
\]

Let \( \beta_2 z = 1 \text{ ns}^2 \) (this value has been demonstrated with Raman amplified DCF fiber)

then

\[
|\delta \omega| = 2 \sqrt{\frac{\pi}{\beta_2 z}} \approx 3.5 \text{ GHz}
\]

\[
|\delta \lambda| \approx 0.028 \text{ nm}
\]
Spectroscopy – a bit of history

- 1666: Newton used a lens and prism to disperse sunlight onto a screen.
- 1812: Fraunhofer developed a diffraction grating.
- 1826: Kirchhoff and Bunsen used gratings to show that each element and compound has its own unique spectrum.
Problems with Traditional Spectrometers

- Optical Spectrometer/Analyzer
  - Bulk optics, complicated, 19th century technology
  - Slow acquisition time, not real time
  - Poor sensitivity

- Detection / Scanning
  - Mechanically rotating grating – slow
  - Focal plan array – not scalable to Mid-IR and Long-IR

Solution: Amplified Dispersive Fourier Transform (UCLA Invention)

- How about the light source?
  - Tunable lasers have limited tuning range
  - Lamps suffer from low power

Solution: femto-second Fiber laser (the ultimate white light source)

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Conventional Spectrometer

- Physically acts like a prism
- Mathematically, it performs Fourier Transform

Wavelength \[\xrightarrow{\text{FT}}\] Space

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Solution:  
**Chirped Wavelength Electronic Encoding and Time domain Sampling (CWEETS)**

- Start with femto-second fiber laser
- Use dispersive fiber or waveguide to perform *Amplified Dispersive Fourier Transform*

Eliminates optical spectrum analyzer – ideal for point of care diagnostics

Femto-second acquisition time => new capability => new scientific discoveries

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Does it really work?

Spectrum of Acetylene

Does it really work?

Spectrum of Acetylene

Carbon Monoxide

Optical spectrum is measured using an electronic circuit!


Live Simulations of the Impact of PMD

Typical PMD value: 0.1-1.0 $\text{ps}/\sqrt{\text{km}}$

Used in these simulations: 15 $\text{ps}/\sqrt{\text{km}}$

References on Dispersive Fourier Transform Spectroscopy and imaging: